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Modern interpretation of Euclid's theory of ratio and proportion

Mark Robert Stecher
Louisiana State University and Agricultural and Mechanical College, bstecher@math.lsu.edu

MODERN INTERPRETATION OF EUCLID'S THEORY OF RATIO AND PROPORTION

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Abstract

Euclid's *Elements* is the foundation for geometry. Book V of Euclid's Elements, which is independent from the earlier books, focuses on multiples, ratios, and proportions. This paper presents a model of the conceptual content of Book V, but using carefully selected modern notation to represent Euclid's ideas without changing them drastically. All of the propositions and proofs from Euclid have been restated using just enough modern language to make clear for a modern reader. We also present a modern theory that bears analogy, proposition by proposition, to Euclid's theory, but uses rigorous modern methods of proof.

Chapter 1. Introduction

The principle aim of this paper is to provide a modern interpretation of Euclid's *Elements of Geometry* Book V. Euclid's Book V concentrates on the development of the theory of ratio and proportion for general magnitudes. The purpose of this modern interpretation is to identify and explain key concepts in Book V in a manner that someone with a reasonably solid background in modern mathematics can follow.

In identifying the key concepts of Book V, it is important to understand the characteristics of early Greek mathematics. Although there are significant differences between ancient Greek mathematics and modern day mathematics, the material covered in a typical high school geometry course mirrors what was taught by Euclid and other Greek mathematicians over two thousand years ago. This paper will add a modern interpretation to Euclid's Book V without changing the logical structure, by making a minimal amount of change to the definitions and propositions in Book V.

In Chapter 2 the necessary background will be presented and key concepts will be introduced, such as characteristics of Greek mathematics, definitions, and propositions from Euclid not found in Book V, but that are necessary to the development of our theory of ratio and proportion.

Perhaps the biggest difference between Euclid's mathematics and modern day mathematics is the absence of numbers from Euclid's geometry. Chapter 2 will discuss this difference and its effect on our modern interpretation. In Book V, Euclid refers to adding and subtracting magnitudes of the same kind. In Chapter 2, the machinery to accomplish this will be given for two types of magnitudes, line segments and polygonal figures. This machinery is established from propositions in Book I and Book II of the *Elements*. For the complex definitions there will be a restatement and a modern restatement, that uses algebraic and modern mathematical notation. This modern restatement of the definition will correspond to the modern restatement of Euclid's propositions that will appear in Chapter 2.

In Chapter 3, the propositions of Book V are stated, restated, and proved in the following format. First there will be Euclid's original formal proposition translated into English. After this we give a restatement, that follows closely Euclid's own restatement known as the "ekthesis". The "ekthesis" is included in every proposition of the elements, directly after the formal statement. It restates what is to be shown by naming objects. The restatement will be changed only minimally to help the reader easily comprehend Euclid's formal statement of the proposition. After the restatement the proof according to Euclid will be given, again with minimal changes. Next there will be a modern restatement, which in some cases is vastly different from the original restatement. The modern restatement will use modern mathematical notation to represent the best modern analogues of Euclid's propositions. Following the modern restatement will be a modern proof. Although some of the modern proofs follow almost exactly the methods used by Euclid, only changing the notation; some propositions are proved using alternative methods, such as induction. It is interesting to note that Euclid's proofs often contain the key step of the formal induction proof. In the formal statement Euclid often states a relation that holds for all multiples, or for any multitude of magnitudes; and his "ekthesis" claims the inductive step, which Euclid proceeds to prove.

Chapter 2. Concepts and Definitions

In this chapter the foundations for a modern model of Euclid's theory of ratio and proportion will be identified. We will discuss the major differences between Greek mathematics and modern day mathematics as it applies to ratio and proportion. We will demonstrate how to apply propositions from Book I and II to Euclid's magnitudes in Book V. In section 2 the key concepts of Book V are summarized and the definitions of Book V are given along with restatements.

2.1 Greek Mathematics

2.1.1 To Arithmetise or Not to Arithmetise

The Greeks worked with magnitudes of various kinds, including line segments, and planar and spatial figures (areas and volumes). Greek mathematicians did not use numbers to measure geometric magnitudes in the way that we today find so natural. Euclid does not refer to the length of a segment, but instead compares line segments directly without mentioning a standard unit.

According to Fowler[5, p. 8], modern mathematics has been "arithmetised". It is characterized by the use of what is now called the number line and its arithmetic. Each segment is assigned a numerical '*length*'. A rectangle is assigned a numerical '*area*' the product of the lengths of the base and height. A 3-dimensional figure has a numerical '*volume*'. In our arithmetised world, ratios are also numbers. For example, the ratio of the circumference of a circle to its diameter has been assigned a number that can be approximated by 3.14 or by the fraction $\frac{22}{7}$.

Although the Greeks may have been able, Greek mathematics is completely non-arithmetised. As Fowler argues, Greek mathematicians confronted directly the objects with which they were concerned. Their geometry dealt with the manipulation of figures. This is evident in the *Elements*, Proposition I.47(also known as the Pythagorean Theorem)

> In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.

Euclid means that a square can be divided into two areas such that the two areas can be manipulated to form two other squares whose sides form the right angle. Modern mathematicians would interpret this result as $c^2 = a^2 + b^2$, where a and b are the sides of the triangle that form the right angle, while c subtends the right angle. The squares have been replaced by their algebraic counterparts the lengths. Greek *arithmētikē* concerned itself ultimately with the evident properties of numbered collections of objects[5, p. 20]. The Greek mathematicians did not use elaborate machinery, instead only natural language[5].

2.1.2 Equality and Comparison

The use of the word 'equal' in the *Elements* is different from our modern use of the word equal. Euclid's 'equality' has a relative meaning, which is best captured by the idea of an equivalence relation. Although Euclid never defines equality, he states axioms about equal objects, such as the Common Notions of Book I[3, p. 155].

Today mathematicians use the word 'equality' to refer to a literal identity. One of the Greek counterparts for 'equal' is the word '*ἴσος*', or isos, which is the origin of the prefix of the word isomorphic. Groups that are alike in every essential way are isomorphic, meaning they have the same group structure, though they may be defined on different sets. The concept of isomorphism parallels Euclid's concept of equality. In many of the propositions, Euclid's use of 'equality' corresponds to the modern use of congruence as it relates to geometrical figures[6, p. 26].

Although there is no definition explicitly given in the *Elements* which states criteria for when line segments are said to be 'equal', in Proposition I.4 two segments are considered 'equal' if they are 'copies' of one another or congruent. Proposition I.1 demonstrates how to construct an equilateral triangle with sides 'equal' to a given segment. The task of constructing an equilateral triangle contains directions on 'copying' a line segment.

In Book I Proposition 4 through Proposition 8 deals with 'equal' angles and 'equal' triangles. For these propositions, the meaning of "equals" is congruent. Again, we think of two angles or two triangles as being 'equal' if they are copies of each other. If two triangles are 'equal' then the corresponding sides which comprise of line segments are 'equal' to one another, as well as the corresponding angles of the two triangles.

Starting in Book I Proposition 35, which states "Parallelograms which are on the same base and in the same parallels are equal to one another with respect to area." Euclid introduces the notion of equality in the sense of area between figures[5, pg. 154]. He begins by adding and subtracting equal (*i.e.*, congruent) triangles to transform one figure into another and concludes that the two figures are "equal", however in this case it is not congruence, but equality of area. This manipulation of the parallelograms is characteristic of mathematics of the time. From his proofs, we can infer that Euclid does not define this new equality, but considers it an undefined notion, like the equality of line segments[5]. In a modern sense, we would call two plane figures equal in area if the same number is assigned to each area. The Greeks dealt directly with the plane figures themselves, by manipulating them into simple shapes, that are easier to compare, such as a square or rectangle. Two plane figures have the same area if one figure can be cut up into finitely many pieces and reassembled into the other solid figure. The converse is not true, a circle having the same 'area' as a square can not be manipulated into a copy of that square.

According to modern usage, when we say that two segments are 'equal' in length we mean that the numbers assigned to them are equal. When two areas are 'equal' we mean the numbers assigned to each are equal. In the case of line segments, a number is assigned by determining a unit length, then using this unit to measure the line segment. Although assigning a number to a plane figure can be done in a similar fashion, it is often determined using properties of the particular plane figure. For example, with a rectangle a number can be assigned to the area by first assigning numbers to the length and width of the rectangle, then multiplying them together. If two line segments have the same number assigned to each of them we say they are congruent. But plane figures must also have the same shape in order to be considered congruent, otherwise the objects only have equal areas.

Today we attach a number to a ratio by taking the quotient of the number that measures the antecedent with the number that measures the consequent. We describe two ratios as being equal if the numbers assigned to each ratio are equal. In Book V, Definition V of Euclid's Elements, Euclid gives a definition that describes when two ratios are the same. As we will see later in the chapter, Euclid has a more complex definition for when two ratios are 'equal'.

In the Elements, magnitudes of the same kind can be compared; they can be 'equal', or they can be greater than or less than one another. The comparison of two line segments can be carried out using the first three propositions of Book I: these enable us to superimpose one line and its endpoint on the other, an operation that is used explicitly in Book I Proposition 4. To compare two plane figures the Greeks would directly compare the figures themselves. If a plane figure can be 'placed' directly onto another solid figure and stay within its bounds then the figure is less than the one it is contained in. However it is not always possible to do this when the plane figures are not equal. If the solid figures are polygons then they can be compared by converting them into squares. Since both plane figures have are squares, either one figure can be placed entirely within the boundaries of the other figure or they are equal.

Notation 2.1.1. We use $M_1 \cong M_2$ to represent that M_1 is 'equal' to M_2, and $M_1 \succ M_2$ to signify that a magnitude M_1 is greater than another magnitude M_2, and $M_1 \prec M_2$ to represent: M_1 is less than M_2.

2.1.3 Adding and Subtracting Magnitudes

Magnitudes of the same kind can be added or a smaller magnitude can be subtracted from a larger magnitude of the same kind. One might add a segment to a segment, or an area to an area. The result would be a new magnitude of the same kind. The propositions in Book V ask us to add and subtract magnitudes, but it does not show us how to do this. However, Euclid's Book I Propositions 1 through 3 demonstrate how to add and subtract line segments, and Proposition I.47 shows us how to add and subtract squares. In order to add line segments we extend one segment and copy the other onto the endpoint so that the copy is superimposed on the extension of the first magnitude. Proposition 1 demonstrates how to construct an equilateral triangle, and in doing so shows us how to copy segments[1, p. 21]. In Proposition I.2 we are shown how to copy the line segment onto the end of another line segment. Finally, the construction in Proposition I.3 gives us a method for adding the segments.

Proposition 2.1.2. (Euclid's Proposition I.2) To place at a given point (as an extremity) a straight line equal to a given straight line.

Proposition 2.1.3. (Euclid's Proposition I.3) Given two unequal straight lines, to cut off from the greater a straight line equal to the less.

Using Propositions from Euclid and the definition below from Hartshorne, we can are given the tools that are required to add magnitudes in the form of line segments.

Modern Definition 2.1.4. Suppose M_1 and M_2 are linear magnitudes, such that $M_1 \cong \overline{AB}$ and $M_2 \cong \overline{BC}$. Then $M_1 + M_2 \cong \overline{AC}$, where segment $\overline{AC}$ is obtained by extending the line segment $\overline{AB}$ in one direction to include point C so that B is strictly between A an C, such that the segment $\overline{BC}$ is congruent to M_2.

Furthermore suppose that $[M_1]$ is the class of all segments equivalent to M_1 or $\overline{AB}$ and $[M_2]$ is the class of all segments equal to line segment M_2 or $\overline{BC}$. Then $[M_1 + M_2]$ is the equivalence class of all segments equal to $\overline{AC}$[7].

From Hilbert[7] and Hartshorne[6], we have the following properties and the corresponding proofs.

Theorem 2.1.5. *Addition of line segment classes has the following properties.*

1. $[M_1 + M_2]$ *is well defined.*

   Proof. Choose a different representative from $[M_1]$, say $A'B'$, extend the line segment $\overline{A'B'}$ to include a point C', such that B' is between A' and C', and so that $\overline{B'C'}$ is a member of $[M_2]$. Then $\overline{A'C'}$ is a member of $[M_1 + M_2]$, and by Common Notion 2: If equals be added to equals then the wholes are equal. Thus $\overline{AC} \cong \overline{A'C'}$.

   ■

2. $[M_1 + M_2] = [M_2 + M_1]$.

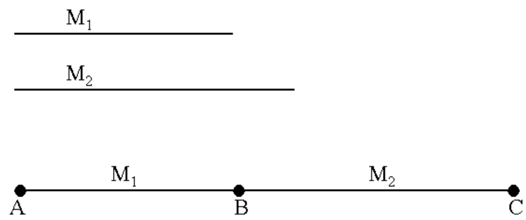

Figure 2.1: Adding Segments

Proof. Let M_2 be the class of all line segments equal to $\overline{BC}$. Extend the line to include the point A, so that B is strictly between A and C, and the segment $\overline{AB}$ is equal to M_1. Thus $M_2 + M_1$ is equal to the line segment $\overline{CA}$. Since the segment $\overline{CA}$ is the same as $\overline{AC}$, then $[M_1 + M_2] = [M_2 + M_1]$.

■

3. $[(M_1 + M_2) + M_3] = [M_1 + (M_2 + M_3)]$.

Proof. As defined above, let $[M_1 + M_2]$ be represented by the segment $\overline{AC}$, extend the line to include point D, so that C is strictly between A and D, and the line segment $\overline{CD}$ belongs to the equivalence class M_3. Then $[(M_1 + M_2) + M_3]$ is represented by the line segment $\overline{AD}$.

Similarly, let $[M_2 + M_3]$ represent the line segment $\overline{BD}$, extend the line to include the point A, so that B is between A and D, so that $\overline{AB}$ is equal to M_1. Thus $\overline{AD}$ is a member of the equivalence class $[M_1 + (M_2 + M_3)]$. Since $\overline{AD}$ is in both equivalence classes $[(M_1 + M_2) + M_3]$ and $[M_1 + (M_2 + M_3)]$, then $[(M_1 + M_2) + M_3] = [M_1 + (M_2 + M_3)]$.

■

In order to consider plane figures in Euclid's theory of ratio and proportion, in addition to the ability to compare two plane figures are equal, we require the ability to add and subtract plane figures. Euclid provides us with a method for adding and subtracting polygons, or as Euclid calls them rectilinear figures. As previously stated, Euclid's Proposition I.47, gives us the ability to add squares such that the 'sum' is a square. In order to add polygons, we cut them up and reassemble them into squares. The first step in the construction is to turn the polygon into a rectangle. This construction is given as Proposition I.45 as a particular case where the given angle is a right angle.

Proposition 2.1.6. (Euclid's Proposition I.45) To construct, in a given rectilineal angle, a parallelogram equal to a given figure.

Euclid's Proposition II.14 demonstrates how to manipulate any polygon into a square.

Proposition 2.1.7. (Euclid's Proposition II.14) To construct a square equal to a rectilineal figure.

Since every polygonal figure can be manipulated into a square of equal area, and squares can be added using the Pythagorean Theorem (Euclid's Proposition I.47), we are able to add any magnitudes that appear in the form of polygons. Turning any polygon into a square will also allow us to determine if any two polygons are equal in area, and if they are not equal which polygon is greater.

Subtracting line segments can be performed using Euclid's Proposition I.3. Subtracting polygons from one another can be accomplished by converting each polygon into a square. After each polygon has been converted into a square Euclid's Proposition I.47 allows us to subtract by making the a side of the larger square the hypotenuse of a right triangle, and the side of the smaller square as one of the legs, then the other remaining side of the triangle will also be the side of the square formed by taking the difference.

Proposition 2.1.8. (Euclid's Proposition I.3) Given two unequal straight lines, to cut off from the greater a straight line equal to the less.

2.2 Euclid Book V

2.2.1 Multiples and Parts

In Book V of Euclid's Elements, Euclid freely formed multiples and parts of magnitudes. Euclid says M_1 measures M_1' when M_1' is obtained by adding to itself some finite number of times. A magnitude is a *multiple* of a smaller magnitude when it is measured by the smaller magnitude without remainder, *i.e.* the larger is composed of a whole number of copies of the smaller magnitude. A magnitude is a *part* of a larger magnitude when it measures that greater magnitude.

When Euclid speaks of one magnitude being a multiple of another, he never creates a symbol to stand for the number that denotes *which* multiple. If we are talking about a multiple of a magnitude M we will call it M' or kM, where k is the number of times that the magnitude M will go into the larger magnitude kM. This notation is absent from Euclid, but it provides a great convenience when we attempt to restate Euclid's theorems.

According to Euclid's usage of the word "multiple", a multiple of a magnitude and the magnitude itself are not equal, *i.e.*, Euclid did not consider the trivial multiple, $k = 1$. The early Greek mathematicians make free use of the cardinal numbers; but rather than our thinking of the sequence $1, 2, 3, \ldots,$ the Greeks would have considered the sequence: duet, trio, quartet, quintet,... as evident in the Greeks *arithmoi*[5]. The unit, or *manos*, has a different status from the others, so that an argument may have to be reformulated when it applies to this case[5, p.13]. However when we say multiple k we are referring to any integer greater than or equal to 1.

Modern Definition 2.2.1. Suppose M is a magnitude and $k \in \{1, 2, 3, \dots\}$

$$kM :\cong \begin{cases} M, & \text{if } k = 1; \\ M + (k-1)M, & \text{if } k > 1. \end{cases}$$

2.2.2 Ratio

According to Book V, Definition 3[4], "a ratio is a sort of relation in respect of size between two magnitudes of the same kind." Since the magnitudes are of the same kind, Euclid directly compared them, thus there was no need for measuring. In modern mathematics, a "ratio" is a function of two variables or quantities, which is similar to the 'ratio' in ancient Greece. However, in modern times a single number is often assigned to the ratio by taking the quotient of the numbers obtained by measuring the antecedent and the consequent. In fact when Euclid does introduce a ratio of two numbers in Book VII, he does not introduce a new definition. Thus a ratio of two numbers is simply a comparison of the two numbers, and is not assigned a single value by taking the quotient. Moreover, there is no evidence that any mathematicians, accountants, or teachers used anything that corresponds to our common fraction.[5, p. 19]

Definition 2.2.2. (Euclid's Definition 4) Magnitudes are said to have a *ratio* to one another which are capable, when multiplied, of exceeding one another.

We will now embark on a piece of modern math that is interesting in its own right, but is not used elsewhere in this thesis.

Modern Definition 2.2.3. Suppose M_1 and M_2 are magnitudes that have a ratio.
$\mathcal{Q}(M_1, M_2) := \{\frac{\ell}{k} \mid kM_1 \succeq \ell M_2\}$

Facts:

1. $q \geq q' \notin \mathcal{Q}(M_1, M_2) \Rightarrow q \notin \mathcal{Q}(M_1, M_2)$.

   Sketch of Proof. Let $q = \frac{s}{t}$ and $q' = \frac{s'}{t'}$. Then $st' \geq s't$ Now.

$$\begin{aligned} q' \notin &\mathcal{Q}(M_1 : M_2), && \text{by hypothesis,} \\ t'M_1 \prec &s'M_2, && \text{by Definition 2.2.3,} \\ st'M_1 \prec &ss'M_2 \\ \frac{s}{t} \geq &\frac{s'}{t'} \end{aligned}$$

$$
\begin{aligned}
s'tM_1 &\prec st'M_1 \\
s'tM_1 &\prec ss'M_2 && \text{transitive property of inequality,} \\
tM_1 &\prec sM_2
\end{aligned}
$$

Therefore $q \notin \mathcal{Q}(M_1, M_2)$. ■

2. $\exists \ell \in \mathbb{N}$ such that $\ell \notin \mathcal{Q}(M_1, M_2)$. By Definition IV.

(1) and (2)imply that $\sup \mathcal{Q}(M_1, M_2)$ exists.

Modern Definition 2.2.4. Suppose M_1 and M_2 are magnitudes that have a ratio, then

$$M_1/M_2 \ := \ \sup \mathcal{Q}(M_1, M_2) \tag{2.2.1}$$

2.2.3 Equality of Ratios

A proportion, *analogon*, is a condition that may or may not hold between four objects. Euclid's gives two definitions of proportionality. In Book V, Definition V is believed to be due to Eudoxos.[5, p. 16] The second in Book VII, Definition XX:

> Numbers are proportional when the first is the same multiple, or the same part, or the same parts, of the second that the third is of the fourth.

For the purposes of developing and interpreting a theory of ratio and proportion, we will be using the first definition.

Definition 2.2.5. Magnitudes are said *to be in the same ratio*, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order.[4]

Modern Definition 2.2.6. *

$$M_1 : M_2 \cong M_3 : M_4 \quad \Longleftrightarrow \quad \forall k, \ell \in \mathbb{N}, \begin{Bmatrix} kM_1 \succ \ell M_2 \Leftrightarrow kM_3 \succ \ell M_4, \\ kM_1 \cong \ell M_2 \Leftrightarrow kM_3 \cong \ell M_4, \\ kM_1 \prec \ell M_2 \Leftrightarrow kM_3 \prec \ell M_4. \end{Bmatrix}$$

Modern Reformulation 2.2.7. **

$$M_1 : M_2 \cong M_3 : M_4 \quad \Longleftrightarrow \quad \{\forall k, \ell \in \mathbb{N}, \ kM_1 \succeq \ell M_2 \Leftrightarrow kM_3 \succeq \ell M_4, \}$$

Modern Proposition 2.2.8. $M_1, \ldots, M_4$ satisfy the right hand side (RHS) of Modern Definition 2.2.6 $\iff$ $M_1, \ldots, M_4$ satisfy the right hand side (RHS) of Modern Reformulation 2.2.7.

$* \Rightarrow **$.
Assume the RHS 2.2.6. Pick k, ℓ.

$$\begin{aligned} kM_1 \succeq \ell M_2 &\Leftrightarrow kM_1 \succ \ell M_2 \text{ or } kM_3 \cong \ell M_4 \\ &\Leftrightarrow kM_3 \succ \ell M_4 \text{ or } kM_3 \cong \ell M_4 \\ &\Leftrightarrow kM_3 \succeq \ell M_4 \end{aligned}$$

■

$** \Rightarrow *$.
Assume RHS 2.2.7. Pick k, ℓ.
i) $kM_1 \succeq \ell M_2 \quad \Leftrightarrow \quad kM_3 \succeq \ell M_4$
ii) $\ell M_1 \succeq kM_2 \quad \Leftrightarrow \quad \ell M_3 \succeq kM_4$
$kM_1 \succ \ell M_2, \Leftrightarrow$ LHS(i) $\wedge$ $\neg$LHS (ii) $\Leftrightarrow$ RHS (i) $\wedge$ $\neg$RHS (ii) $\Leftrightarrow$ $kM_3 \succ \ell M_4$.
$kM_1 \cong \ell M_2, \Leftrightarrow$ LHS(i) $\wedge$ LHS (ii) $\Leftrightarrow$ RHS (i) $\wedge$ RHS (ii) $\Leftrightarrow$ $kM_3 \cong \ell M_4$.
$kM_1 > \ell M_2, \Leftrightarrow$ $\neg$LHS(i) $\wedge$ LHS (ii) $\Leftrightarrow$ $\neg$ RHS (i) $\wedge$ RHS (ii) $\Leftrightarrow$ $kM_3 > \ell M_4$. ■

Theorem 2.2.9. *Any two magnitudes that have a ratio, determine a real number, in other words,*

1. $\mathcal{Q}(M_1, M_2)$ *depends only on the ratio.*
2. $M_1 : M_2 \cong (M_1/M_2) : 1$.

Proof.

1. This is obvious.
2. $kM_1 \succeq \ell M_2 \Leftrightarrow \frac{\ell}{k} \in \mathcal{Q}(M_1 : M_2) \Leftrightarrow \frac{\ell}{k} \leq M_1/M_2 \Leftrightarrow k(M_1/M_2) \succeq \ell(1)$.

■

2.2.4 A Common Hypothesis

Book V of Euclid is independent from the other books in Euclid's Elements. A hypothesis that occurs frequently in Euclid V is if "a first magnitude is the same multiple of a second, that a third magnitude is of the fourth." This hypothesis appears in Proposition II, Proposition III, and Proposition VI of Book V. Propositions IV, V, XIII, XIV, and XXIV have very nearly the same hypothesis as mentioned above. In Proposition II, we have the additional hypothesis that the fifth is the same multiple of the second as the sixth is of the fourth.

Chapter 3. Propositions of Book V

In the *Elements* Euclid states the proposition in a formal way, then restates what is given in terms of particular objects, and what results are sought in relation to the particular objects. For example, Euclid's formal statement for Proposition I.1: On a given finite straight line to construct an equilateral triangle. Before the proof, Euclid restates the proposition using a particular case: Let AB be the given finite straight line, and what is sought: Thus it is required to construct an equilateral triangle on the straight line AB.

3.1 Proposition I

Euclid's Statement

If there be any number of magnitudes whatever which are, respectively, equimultiples of any magnitudes equal in multitude, then, whatever multiple one of the magnitudes is of one, that multiple also will be of all.

Restatement

Suppose $M_1, \ldots, M_n$ and $M'_1 \ldots M'_n$ are magnitudes, and suppose each M'_i is the same multiple of M_i. Then $(M'_1 + \cdots + M'_n)$ is that same multiple of $(M_1 + \cdots + M_n)$.

Proof following Euclid. Divide M'_1 into magnitudes or parts equal to M_1. Similarly divide M'_2 into parts equal to M_2. Since M'_1 and M'_2 are equimultiples of M_1 and M_2, then the number of parts of M'_1 is the same as the number of parts of M'_2.

Since each part of M'_1 is equal to M_1 and each part of M'_2 is equal to M_2 then the sum of any one part of M'_1 and any one part of M'_2 is equal to $M_1 + M_2$.

Thus if we choose another part from M'_1 and a different part of M'_2, the of these parts will also be equal to $M_1 + M_2$. There are the same number of parts in each of M'_1 and M'_2. Therefore the of parts in M'_1 equal to M_1 is equal to the number of parts of $M'_1 + M'_2$ equal to $M_1 + M_2$.

Therefore the sum of any multitude of magnitudes that are equimultiples of other magnitudes is equal to the multiple of the sum of lesser magnitudes. ■

Modern Restatement

For any positive integers k, ℓ and for any magnitudes $\{M_1, \ldots, M_n\}$,

$$kM_1 + \cdots + kM_n \cong k\,(M_1 + \cdots + M_n)$$

Modern Proof. Let $\Theta\,(n, k)$ be the statement:

$$\text{for any magnitudes}\,\{M_1, \ldots, M_n,\}\,,\ kM_1 + \cdots + kM_n \cong k\,(M_1 + \cdots + M_n)$$

We show $\Theta\,(n, k)$ by induction.
1) Claim: $\forall k\ \Theta\,(1, k)$. This is obvious.
2) Claim: $\forall k\ \Theta\,(2, k)$. This will be proved by induction on k. $\Theta\,(2, 1)$ is obvious. We now prove that $\Theta\,(2, k-1) \Rightarrow \Theta\,(2, k)$. Let $k > 1$ be an arbitrary whole number. Then

$$\begin{aligned}
kM_1 + kM_2 &\cong (M_1 + (k-1)\,M_1) + (M_2 + (k-1)\,M_2), && \text{by Definition 1,}\\
&\cong M_1 + M_2 + (k-1)\,M_1 + (k-1)\,M_2, && \text{Property of whole numbers,}\\
&\cong M_1 + M_2 + (k-1)\,(M_1 + M_2), && \theta(2, k-1),\\
&\cong k\,(M_1 + M_2), && \text{by Definition 1.}
\end{aligned}$$

3) Claim: $[\forall k\ \ \Theta\,(n-1, k)] \Rightarrow [\forall k\ \ \Theta\,(n, k)]$
Assume $\forall k\ \ \Theta\,(n-1, k)$ and fix k.

$$\begin{aligned}
kM_1 + \cdots + kM_{n-1} + kM_n &\cong k\,(M_1 + \cdots + M_{n-1}) + kM_n, && \text{by } \theta\,(n-1, k),\\
&\cong k\,(M_1 + \cdots + M_n), && \text{by } \theta\,(2, k).
\end{aligned}$$

■

Commentary

Euclid, of course did not use induction in his form of the proof. Euclid's proof concerns the particular case where $n = 2$, but includes an "and-so-on".

3.2 Proposition II

Euclid's Statement

If a first magnitude be the same multiple of a second that a third is of a fourth, and a fifth also be the same multiple of the second that a sixth is of the fourth, the sum of the first and fifth will also be the same multiple of the second that the sum of the third and sixth is of the fourth.

Restatement

Let a first magnitude M_1' be the same multiple of M_1 that M_2' is of M_2. Also let M_1'' be the same multiple of M_1 that M_2'' is of M_2. Then the sum $M_1' + M_1''$ will be the same multiple of M_1 that the sum $M_2' + M_2''$ is of M_2.

Proof following Euclid. Divide M_1' into parts equal to M_1. Similarly, divide M_2' into parts equal to M_2. The number of magnitudes equal to M_1 of M_1' is the same as the number of magnitudes equal to M_2 of M_2', because they are equimultiples of M_1 and M_2 respectively.

For the same reason, the number of parts equal to M_1 in M_1'' is equal to the number of parts equal to M_2 in M_2''.

Therefore there is an equal number of parts equal to M_1 in $M_1' + M_1''$ as there are parts in $M_2' + M_2''$ equal to M_2. Thus $M_1' + M_1''$ and $M_2' + M_2''$ are equimultiples of M_1 and M_2 respectively. Thus whatever multiple $M_1' + M_1''$ is of M_1, is the same as the multiple that $M_2' + M_2''$ is of M_2.

■

Modern Restatement

For any positive integers k, ℓ and for any magnitude M,

$$kM + \ell M \cong (k+\ell)M \quad \forall M$$

.

Modern Proof. We use induction. Fix M and let $\Theta(k,\ell)$ be the statement $(k+\ell)M \cong kM + \ell M$.

1) Claim: $\Theta(1,1)$ is obvious.

2) Claim: $\Theta(1,\ell-1) \Rightarrow \Theta(1,\ell)$

$$\begin{aligned}(1+\ell)M &\cong M + ((\ell+1)-1)M, && \text{by Definition 2.2.1,}\\ &\cong 1M + \ell M, && \text{by Definition 2.2.1.}\end{aligned}$$

3) Claim: $\Theta(k-1,\ell-1) \Rightarrow \Theta(k,\ell)$

$$\begin{aligned}(k+\ell)M &\cong M + (k+\ell-1)M, && \text{by Definition 2.2.1,}\\ &\cong M + M + (k+\ell-2)M, && \text{by Definition 2.2.1,}\\ &\cong M + M + ((k-1)+(\ell-1))M, && \text{by associativity,}\\ &\cong M + M + (k-1)M + (\ell-1)M, && \text{by } \Theta(k-1,\ell-1)\\ &\cong M + (k-1)M + M(\ell-1)M, && \text{by associativity,}\\ &\cong kM + \ell M, && \text{by Definition 2.2.1.}\end{aligned}$$

■

Commentary

We will show that equality also holds for any finite number of multiples being added.

Let $\Phi(n)$ be the statement $(k_1 + \cdots + k_n)M \cong k_1M + \cdots + k_nM$.

Proof. 1)Claim: $\Phi(1)$ is obvious.
2)Claim: $\Phi(2)$ was proved.
3)Claim: $\Phi(n-1) \Rightarrow \Phi(n)$.

$$\begin{aligned}
(k_1 + \cdots + k_n)M &\cong ((k_1 + \cdots + k_{n-1}) + k_n)M, && \text{by associativity,} \\
&\cong (k_1 + \cdots + k_{n-1})M + k_nM, && \text{by } \Phi(2), \\
&\cong k_1M + \cdots + k_{n-1}M + k_nM, && \text{by } \Phi(n-1).
\end{aligned}$$

Therefore $\Phi(n) \quad \forall n \in \mathbb{N}$.

■

3.3 Proposition III

Euclid's Statement

If a first magnitude be the same multiple of a second that a third is of a fourth and if equimultiples be taken of the first and third then also the magnitudes taken will be equimultiples respectively, the one of the second and the other of the fourth.

Restatement

Let a first magnitude M_1' be the same multiple of M_1 that M_2' is of M_2. Also let M_1'' and M_2'' be equimultiples of M_1' and M_2'. Then M_1'' and M_2'' are equimultiples of M_1 and M_2.

Proof following Euclid. Since M_1'' and M_2'' are equimultiples of M_1' and M_2' respectively, then there is the same number of parts in M_1'' equal to M_1' as there are parts in M_2'' equal to M_2'.

Divide M_1'' into parts equal to M_1', and divide M_2'' into parts equal to M_2'. Since M_1'' and M_2'' are equimultiples of M_1' and M_2', then the magnitude M_1'' can be expressed as the sum $M_1' + \cdots + M1'$, and M_2'' as the sum $M_2' + \cdots + M2'$.

But since each M_1' in the sum and each M_2' in the sum are equimultiples of M_1 and M_2 then M_1'' and M_2'' are being expressed as sums with the same number of terms, in which corresponding terms are equimultiples, since all terms are the same multiple of M_1 and M_2. Thus by Proposition II, M_1'' and M_2'' are equimultiples of M_1 and M_2.

■

Modern Restatement

For any positive integers k, ℓ and for any magnitude M, $k(\ell)M \cong (k\ell)M]$

Modern Proof. Let $\Theta(\ell)$ be the statement: $[\forall k{\in}N \quad k(\ell)M \cong (k\ell)M]$.
We will show $\Theta(\ell)$ holds for all $\ell \in \mathbb{N}$ by induction.
1)Claim: $\Theta(1)$ is obvious.
2)Claim: $\Theta(\ell) \Rightarrow \Theta(\ell+1)$.

$$\begin{aligned} k((\ell)M) \cong & k(M+(\ell)M) && \text{by Definition 2.2.1,} \\ & kM + k(\ell M) && \text{by Proposition I,} \\ & kM + (k\ell)M && \text{by } \Theta(\ell), \\ & (k+k\ell)M && \text{by Proposition II,} \\ & (k(\ell+1))M && \text{by arithmetic.} \end{aligned}$$

Therefore $\Theta(\ell) \quad \forall \ell \in \mathbb{N}$.

■

3.4 Proposition IV

Euclid's Statement

If a first magnitude have to a second the same ratio as a third to a fourth, any equimultiples whatever of the first and third will also have the same ratio to any equimultiples whatever of the second and the fourth respectively, taken in corresponding order.

Restatement

If a first magnitude M_1 has to a second magnitude M_2 the same ratio as a third M_3 has to a fourth magnitude M_4. Also if equimultiples M_1' and M_3' be taken of M_1 and M_3, and equimultiples M_2' and M_4' be taken of M_2 and M_4. Then M_1' has to M_2' the same ratio as M_3' has to a M_4'.

Proof following Euclid. Take equimultiples M_1'' and M_3'' of M_1' and M_3'. Also take equimultiples M_2'' and M_4'' of M_2' and M_4'.

Since M_1' and M_3' are equimultiples of M_1 and M_3, and since equimultiples M_1'' and M_3'' of M_1' and M_3' were taken, then M_1'' and M_3'' are equimultiples of M_1 and M_3.

Similarly, M_2'' and M_4'' are equimultiples of M_2 and M_4.

Since equimultiples M_1'' and M_3'' of M_1 and M_3, and equimultiples M_2'' and M_2'' of M_2 and M_4 have been taken; and M_1 is to M_2 is the same ratio as M_3 is to M_4;

therefore if M''_1 is in excess of M''_2, then M''_3 is in excess of M''_4; if M''_1 is equal to M''_2, then M''_3 is equal to M''_4; if M''_1 is less than M''_2, then M''_3 is less than M''_4.

Since M''_1 and M''_3 are equimultiples of M'_1 and M'_3, and M''_2 and M''_4 are any equimultiples of M'_2 and M'_4, then M'_1 is to M'_2 as M'_3 is to M'_4. ■

Modern Restatement

Suppose $M_1 : M_2 \cong M_3 : M_4$. Then for all positive integers ℓ, t $\ell M_1 : tM_2 \cong \ell M_3 : tM_4$.

Modern Proof. Suppose $M_1 : M_2 \cong M_3 : M_4$. Fix positive integers ℓ, t. Then for all positive integers k, s;

$$\begin{array}{lcccl} & k\ell\,(M_1) \succeq st\,(M_2) & \Leftrightarrow & k\ell\,(M_3) \succeq st\,(M_4) & \text{by Definition 2.2.5.} \\ \text{Thus } \forall k, s & k\,(\ell M_1) \succeq s\,(tM_2) & \Leftrightarrow & k\,(\ell M_3) \succeq s\,(tM_4) & \text{by Proposition III.} \\ \text{Thus} & \ell M_1 : tM_2 \cong \ell M_3 : tM_4 & & & \text{by Definition 2.2.5.} \end{array}$$

■

3.5 Proposition V

Euclid's Statement

If a magnitude be the same multiple of a magnitude that a part subtracted is of a part subtracted, the remainder will also be the same multiple of the remainder that the whole is of the whole.

Restatement

Let M_2 be the whole and M_1 a part subtracted. Suppose M'_1 and M'_2 are equimultiples of M_1 and M_2, then the remainder of $M'_2 - M'_1$ is the same multiple of the remainder of $M_2 - M_1$ as M'_2 is of M_2.

Proof following Euclid. Whatever multiple M'_1 is of M_1, let $M'_2 - M'_1$ be that same multiple of another magnitude M_3. Thus M'_1 and $M'_2 - M'_1$ are equimultiles of M_1 and M_3.

By Proposition I, M'_2 which is equal to $M'_1 + (M'_2 - M'_1)$ is the same multiple of $M_3 + M_1$ as M'_1 is of M_1. But by the hypothesis, M'_1 and M'_2 are equimultiples of M_1 and M_2. Thus since M'_2 is the same multiple of M_2 and $M_1 + M_3$, then M_2 is equal to $M_1 + M_3$.

Let M_1 be subtracted from each magnitude M_2 and $M_1 + M_3$. Therefore the remainder M_3 is equal to $M_2 - M_1$.

Since M'_1 and $M'_2 - M'_1$ are equimultiples of M_1 and M_3 respectively, and because M_3 is equal to $M_2 - M_1$, thus M'_1 is the same multiple of M_1 that $M'_2 - M'_1$ is of $M_2 - M_1$.

Also M_1' is the same multiple of M_1 that M_2' is of M_2; therefore $M_2' - M_1'$ is the same multiple of $M_2 - M_1$ that M_2' is of M_2.

In other words the remainder $M_2' - M_1'$ is the same multiple of the remainder $M_2 - M_1$ that the whole magnitude M_2' is of M_2.

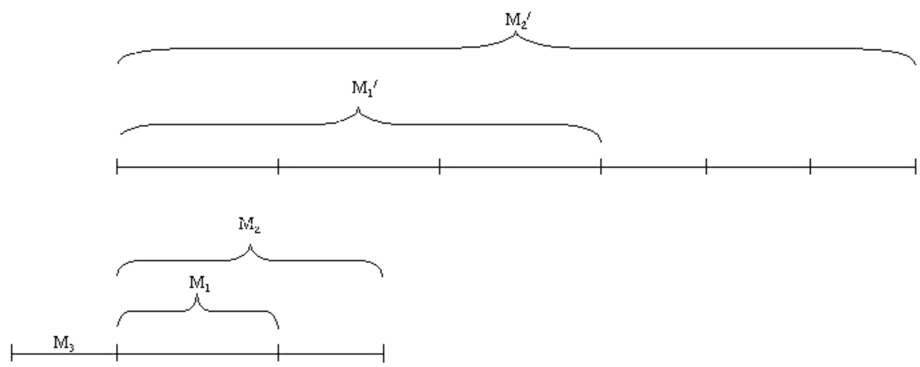

Figure 3.2: Proposition 5

■

Modern Restatement

If M_2 is the magnitude and M_1 a smaller magnitude, then $k(M_2 - M_1) \cong kM_2 - kM_1 \ \forall k \in \mathbb{N}$.

Modern Proof. Let $\Theta(k)$ be the statement $k(M_2 - M_1) \cong kM_2 - kM_1$. We show by induction that $\Theta(k)$ holds $\forall k \in \mathbb{N}$.
1)Claim: $\Theta(1)$.

$$\begin{aligned} 1(M_2 - M_1) &\cong M_2 - M_1 && \text{by Definition 2.2.1,} \\ &\cong 1M_2 - 1M_1 && \text{by Definition 2.2.1.} \end{aligned}$$

2)Claim: $\Theta(k-1) \Rightarrow \Theta(k)$.

$$
\begin{aligned}
k(M_2 - M_1) \cong & (M_2 - M_1) + (k-1)(M_2 - M_1) && \text{by Definition 2.2.1,} \\
\cong & (M_2 - M_1) + (k-1)M_2 - (k-1)M_1 && \text{by} \Theta\left(k-1\right), \\
\cong & M_2 + (k-1)M_2 - M_1 - (k-1)M_1 && \text{by the commutative property,} \\
\cong & (M_2 + (k-1)M_2) - (M_1 + (k-1)M_1) && \text{by arithmetic,} \\
\cong & kM_2 - kM_1 && \text{by Definition 2.2.1.}
\end{aligned}
$$

Therefore $\Theta\left(k\right) \forall k \in \mathbb{N}$.

■

Commentary

This proposition corresponds to Proposition I where subtraction replaces addition.

3.6 Proposition VI

This Proposition is similar to Proposition II from Book V with subtracting replacing addition.

Euclid's Statement

If two magnitudes be equimultiples of two magnitudes, and any magnitudes subtracted from them be equimultiples of the same, the remainders also are either equal to the same or equimultiples of them.

Restatement

Let two magnitudes M_1' and M_2' be equimultiples of magnitudes M_1 and M_2. Also let M_1'' and M_2'' be lesser multiples of M_1 and M_2. Then the remainders $M_1' - M_1''$ and $M_2' - M_2''$ formed by subtracting the lesser multiples from the greater multiples, will either equal M_1 and M_2, or be equimultiples of M_1 and M_2.

Proof following Euclid. Since M_1' is a multiple of M_1, and M_1'' is a lesser multiple of M_1, then $M_1' - M_1''$ is either M_1 or a multiple of M_1.

Assume that $M_1' - M_1''$ is not a multiple of M_1 and $M_1' - M_1''$ is not equal to M_1.

Then $M_1' - M_1''$ could not be divided into parts equal to M_1.

If $M_1' - M_1''$ is smaller than M_1, then $M_1'' + (M_1' - M_1'')$ cannot be divided into parts equal to M_1, because if we subtract magnitudes equal to M_1 from $M_1'' + (M_1' - M_1'')$ we will be able to subtract the same number that divide M_1'', which

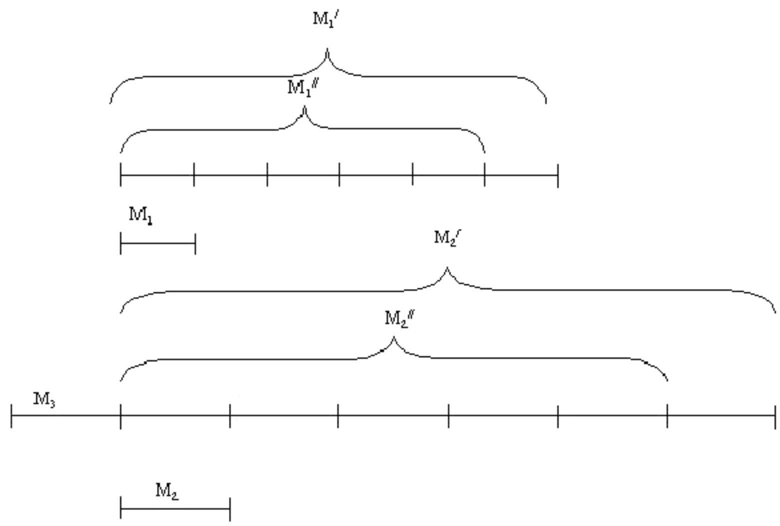

Figure 3.3: Proposition 6

will leave a remainder equal to $(M_1' - M_1'')$. But M_1' is equal to $M_1'' + (M_1' - M_1'')$, thus M_1' can not be divided into parts equal to M_1 which is a contradiction.

If $M_1' - M_1''$ is larger than M_1, then subtract out as many magnitudes equal to M_1 as possible from $M_1' - M_1''$ and call the remainder M_3.

The sum of all of the magnitudes subtracted $(M_1' - M_1'') - M_3$ is a multiple of M_1 and M_1'' is also a multiple of M_1.

The sum $[(M_1' - M_1'') - M_3] + M_1''$ would be a multiple of M_1 by Prop II. But M_1' is equal to $[(M_1' - M_1'') - M_3] + M_1'' + M_3$ would not be a multiple, which is also a contradiction.

Let $M_1' - M_1''$ be equal to M_1. Then $M_2' - M_2''$ is equal to M_2.

> Add a magnitude M_3 to the magnitude M_2'', such that M_3 is equal to M_2. Since M_1'' is the same multiple of M_1 that M_2'' is of M_2, and since $M_1' - M1''$ is equal to M_1 and M_3 is equal to M_2, therefore M_1' is the same multiple of M_1 that $M_3 + M_2''$ is of M_2, by Proposition II. At the same time, M_1' is the same multiple of M_1 that M_2' is of M_2, thus $M_3 + M_2''$ is the same multiple of M_2 that M_2' is of M_2. Hence M_2' is equal to $M_3 + M_2''$. Subtract the magnitude M_2'' from each magnitude M_2' and $M_2'' + M_3$, and the corresponding remainders $M_2' - M_2''$ and M_3 are equal. Since M_3 is equal to M_2, then $M_2' - M_2''$ is also equal to M_2.
>
> Therefore if $M_1' - M_1''$ is equal to M_1, then $M_2' - M_2''$ is equal to M_2.

Similarly we can prove that if $M_1' - M_1''$ is a multiple of M_1, then $M_2' - M_2''$ is the same multiple of M_2.

> Begin by adding a magnitude M_3 to the magnitude M_2'', such that M_3 is the same multiple of M_2 that $M_1' - M_1''$ is of M_1. Since M_1'' is the same multiple of M_1 that M_2'' is of M_2, and at the same time $M_1' - M1''$ is the same multiple of M_1 as M_3 is to M_2, then M_1' is equal to $M_1'' + (M_1' - M_1'')$ is the same multiple of M_1 that $M_3 + M_2''$ is of M_2, by Proposition II. Also since M_1' is the same multiple of M_1 that M_2' is of M_2, then $M_3 + M_2''$ and M_2' are equimtultiples of the same magnitude M_2. Therefore M_2' is equal to $M_3 + M_2''$. Subtract the magnitude M_2'' from each magnitude M_2' and $M_2'' + M_3$, and the corresponding remainders $M_2' - M_2''$ and M_3 are equal. Since M_3 is the same multiple of M_2 that $M_1' - M_1''$ is of M_1, then $M_2' - M_2''$ is the same multiple of M_2 that $M_1' - M_1''$ is of M_1.

■

Modern Restatement

Let M be a magnitude, and $k, \ell \in \mathbb{N}$, then $(k - \ell)\, M \cong kM - \ell M$

Modern Proof. Fix M and let $\Theta\,(k, \ell)\, M$ be the statement $(k - \ell)\, M \cong kM - \ell M$.
1) $\Theta\,(2, 1)$ is obvious.
2) $\Theta\,(k - 1, 1) \Rightarrow \Theta\,(k, 1)$

$$\begin{array}{rcll} (k-1)M & \cong & M + ((k-1) - 1)\, M, & \text{by Definition 2.2.1,} \\ & \cong & M + (k-1)\, M - 1M, & \text{by } \Theta\,(k-1, 1), \\ & \cong & kM - 1M, & \text{by Definition 2.2.1.} \end{array}$$

3) Claim: $\Theta\,(k - 1, \ell - 1) \Rightarrow \Theta\,(k, \ell)$

$$\begin{array}{rcll} (k-\ell)M & \cong & (k - \ell - 1 + 1)M, & \text{property of addition,} \\ & \cong & ((k-1) - (\ell - 1))M, & \text{by associativity,} \\ & \cong & (k-1)M - (\ell-1)M, & \text{by } \Theta\,(k-1, \ell-1), \\ & \cong & (k-1)M - (\ell-1)M - M + M, & \text{property of addition,} \\ & \cong & (M + (k-1)M) - (M + (\ell-1)M), & \text{by associativity,} \\ & \cong & kM - \ell M, & \text{by Definition 2.2.1.} \end{array}$$

Therefore $(k - \ell)M \cong kM - \ell M \qquad \forall\, k, l \in \mathbb{N}$.

■

3.7 Proposition VII

Euclid's Statement

Equal magnitudes have to the same the same ratio, as also has the same to equal magnitudes.

Restatement

Suppose M_1, M_2, and M_3 be magnitudes of the same kind, such that M_1 is equal to M_2. Then M_1 is to M_3 the same ratio as M_2 is to M_3 and M_3 is to M_1 the same ratio as M_3 is to M_2.

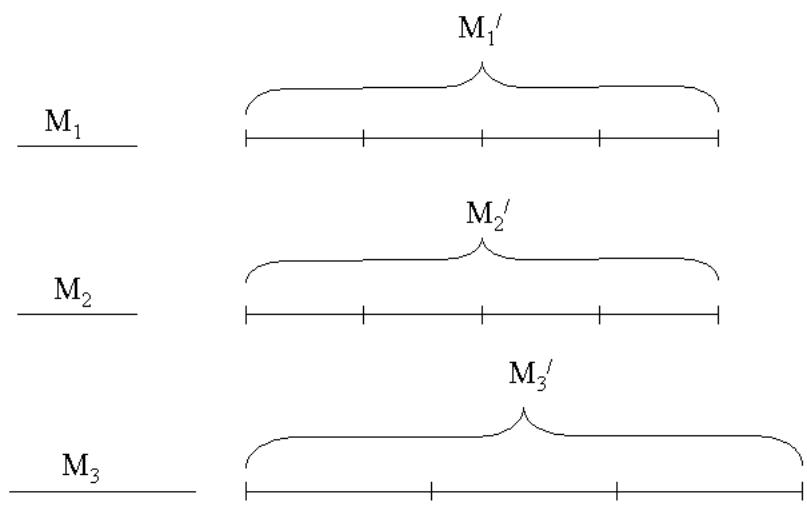

Figure 3.4: Proposition 7

Proof following Euclid. Let equimultiples M_1' and M_2' be taken of M_1 and M_2, and let M_3' be any multiple of M_3. Since M_1 is equalt o M_2, then M_1' is equal to M_2'.

If M_1' is greater than M_3', then M_2' is greater than M_3'. If M_1' is less than M_3', then M_2' is less than M_3'. If M_1' is equal to M_3', then M_2' is equal to M_3'.

Also, M_1' and M_2' are equimultiles of M_1 and M_2, and M_3' is any multiple of M_3.

Therefore M_1 is to M_3 the same ratio as M_2 is to M_3 by definition of ratio.

Similarly, if M_3' is greater than M_1', then M_3' is greater than M_2'. If M_3' is less than M_1', then M_3' is less than M_2'. If M_3' is equal to M_1', then M_3' is equal to M_2'.

At the same time M_3' is a multiple of M_3, while M_1' and M_2' are equimultiles of M_1 and M_2.

Then also M_3 is to M_1 as M_3 is to M_2 by definition of ratio. ■

Modern Restatement

Suppose $M_1 \cong M_2$. Then $M_1 : M_3 \cong M_2 : M_3$ and $M_3 : M_1 \cong M_3 : M_2$.

Modern Proof. For any positive integer k, $kM_1 \cong kM_2$.

Thus $kM_1 \succeq \ell M_3 \Leftrightarrow kM_2 \succeq \ell M_3$. Therefore $M_1 : M_3 \cong M_2 : M_3$.

Similarly, $\ell M_3 \succeq kM_1 \Leftrightarrow \ell M_3 \succeq kM_2$. Therefore $M_3 : M_1 \cong M_3 : M_2$. ■

Corollary 3.7.1. *If any magnitudes are proportional then they are also proportional inversely. In other words, if $M_1 : M_2 \cong M_3 : M_4$, then $M_2 : M_1 \cong M_4 : M_3$.*

Proof. Let kM_1 and kM_3 be equimultiples of M_1 and M_3. Also let ℓM_2 and ℓM_4 be equimultiples of M_2 and M_4. Thus

$$kM_1 \succeq \ell M_2, \qquad \Longleftrightarrow \qquad kM$$

Thus

$$\ell M_2 \succeq kM_1, \qquad \Longleftrightarrow \qquad \ell M$$

Therefore $M_2 : M_1 \cong M_4 : M_3$.

■

Commentary

This Corollary is given as a "Porism". It is not Euclid's habit to explain a Porism, since a Porism should be a by-product appearing without effort or trouble.[4, p. 174] Aristotle assumes inversion in *Meteorologica III.*[4, p. 149]

3.8 Proposition VIII

Definition 3.8.1. When, of the equimultiples, the multiple of the first exceeds the multiple of the second, but the multiple of the third does not exceed the multiple of the fourth, then the first is said to *have a greater ratio* to the second than the third has to the fourth.[4, p.114]

Modern Definition 3.8.2. For any numbers k and ℓ such that

$$kM_1 \succ \ell M_2$$

and

$$kM_3 \preceq \ell M_4,$$

then we say

$$M_1 : M_2 \succ M_3 : M_4$$

Euclid's Statement

Of unequal magnitudes, the greater has to the same a greater ratio than the less has; and the same has to the less a greater ratio than it has to the greater.

Restatement

Let M_1 and M_2 be unequal magnitudes, and let M_1 be greater. Also let M_3 be a magnitude of any size. Then M_1 has to M_3 a greater ratio than M_2 has to M_3.

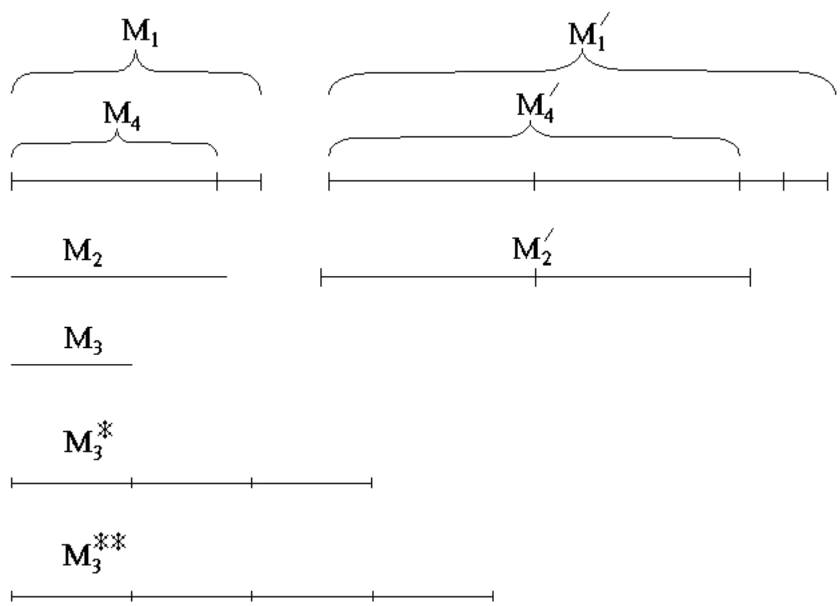

Figure 3.5: Proposition 8 Case I

Proof following Euclid. Since M_1 is greater than M_2, consider a portion of M_1 equal to M_2 and name it M_4.

Then the smaller of the magnitudes M_4, $M_1 - M_4$, if multiplied will eventually be greater than M_3 by definition 4. Case I will assume $M_1 - M_4$ is smaller than M_4, while Case II will assume that M_4 is smaller than $M_1 - M_4$.

Case I

> Let $M_1 - M_4$ be smaller than M_4, and let $(M_1 - M_4)'$ be a multiple of $M_1 - M_4$ that is greater than M_3, then take that same multiple M_4' and M_2' of M_4 and M_2.
>
> Let M_3^* be the greatest multiple of M_3 that is less than or equal to M_2'. Then the very next multiple of M_3 will be the smallest multiple that is greater than M_2', call it M_3^{**}.
>
> Thus M_2' is greater than or equal to M_3^* and M_2' is smaller than M_3^{**}.
>
> Since $(M_1 - M_4)'$ is the same multiple of $M_1 - M_4$ that M_4' is of M_4, then $(M_1 - M_4)'$ is the same multiple of $M_1 - M_4$ that $(M_1 - M_4)' + M_4'$ is of M_1 by Proposition I.
>
> Now since $(M_1 - M_4)' + M_4'$ is a multiple of M_1 we will call it M_1'. Thus we can also refer to $(M_1 - M_4)'$ as $M_1' - M_4'$.
>
> At the same time $M_1' - M_4'$ is the same multiple of $M_1 - M_4$ that M_2' is of M_2. Thus M_1' is the same multiple of M_1 that M_2' is of M_2. Therefore M_1' and M_2' are equimultiples of M_1 and M_2.
>
> Also recall that M_4' is the same multiple of M_4 that M_2' is of M_2, and since M_4 was taken to be equal to M_2, then M_4' is equal to M_2'.
>
> Thus because M_2' is greater than or equal to M_3', then so M_4' is also greater than or equal to M_3^*.
>
> Also $M_1' - M_4'$ is greater than M_3, therefore M_1' is greater than $M_3 + M_3^*$.
>
> But $M_3 + M_3^*$ is equal to M_3^{**}, thus $M1'$ is greater than M_3^{**}. So $M1'$ is greater than M_3^{**} while M_2' is less than M_3^{**}.
>
> Also M_1' and M_2' are equimultiples of M_1 and M_2, and at the same time M_3'' is some multiple of M_3, therefore M_1 has to M_3 a greater ratio than M_2 has to M_3 by Proposition VII.
>
> Similarly we can show that M_3 has to M_2 a greater ratio than M_3 has to M_1. Because if M_3^{**} is greater than M_2' while M_3^{**} is not greater than M_1', and if M_3^{**} a multiple of M_3, while M_1' and M_2' equimultiples of M_1 and M_2, then M_3 has to M_2 a greater ratio than M_3 has to M_1.

Case II

> Now let M_4 be smaller than $M_1 - M_4$. Also by Definition 4, the smaller magnitude, M_4, will eventually be greater than M_3.

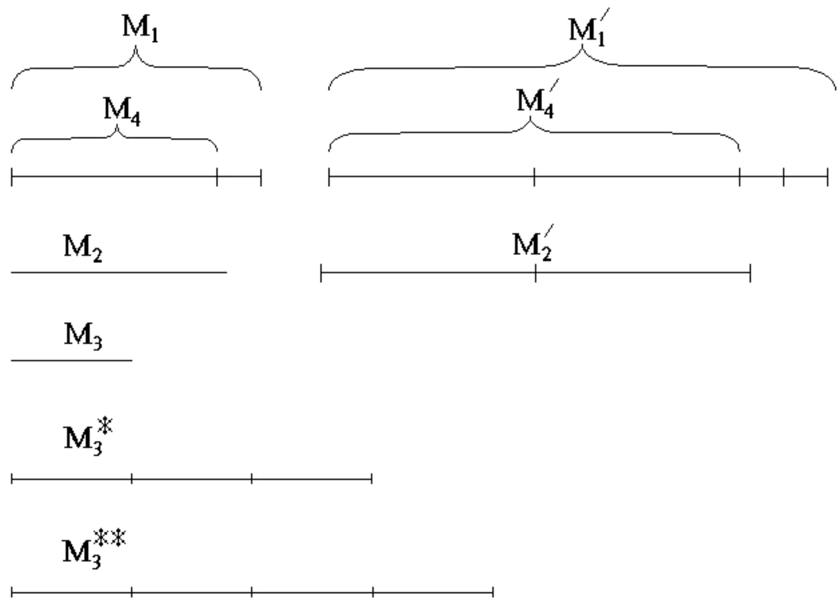

Figure 3.6: Proposition 8 Case II

Let M_4' be a multiple of M_4 so that M_4' is greater than M_3, and what ever multiple M_4' is of M_4 take that same multiple $M_1' - M_4'$ of $M_1 - M_4$ and also M_2' of M_2.

Then show that M_1' and M_2' are equimultiples of M_1 and M_2. Since $(M_1 - M_4)'$ and M_4 are equimultiples of $M_1 - M_4$ and M_4, then the whole $(M_1 - M_4)' + M_4'$ and M_4' are equimultiples of $(M_1 - M_4) + M_4$ and M_4. But $(M_1 - M_4) + M_4$ is the same as M_1, so $(M_1 - M_4)' + M_4'$ and M_4' are equimultiples of M_1 and M_4.

Since $(M_1 - M_4)' + M_4'$ is a multiple of M_1 we will rename this magnitude as M_1'. This allows us to refer to $(M_1 - M_4)'$ as $M_1' - M_4'$.

So M_1' and M_4' are equimultiples of M_1 and M_4, and also M_2' and M_4' are equimultiples of M_2 and M_4. Thus M_1' and M_2' are equimultiples of M_1 and M_2.

Again let M_3^* be the greatest multiple of M_3 less than or equal to $M_1' - M_4'$ and let M_3^{**} be the smallest multiple of M_3 that is greater than $M_1' - M_4'$. Thus M_3^{**} is one multiple more of M_3 than M_3^* is of M_3. In other words $M_3^* + M_3$ is equal to M_3^{**}.

Because M_4' is greater than M_3 and because $M_3^*$is the greatest multiple of M_3 less than or equal to $M_1' - M_4'$, then $M_4' + (M_1' - M_4')$ is greater than $M_3 + M_3^*$. In other words M_1' is greater than M_3^{**}.

Now show that M_2' is not greater than M_3^*. Recall that Case II began with the assumption that M_4 is less than or equal to $M_1 - M_4$. Also M_4 is equal to M_2. Thus M_2 is less than or equal to $M_1 - M_4$. Also since equimultiples M_2' and $(M_1 - M_4)'$ which is equal to $M_1' - M_4'$ of M_2 and $M_1 - M_4$ were taken, thus M_2' is less than or equal to $M_1' - M_4'$. And recall that we chose M_3^{**} so that $M_1' - M_4'$ is less than M_3^{**}. Since M_2' is less than or equal to $M_1' - M_4'$ and $M_1' - M_4'$ is less than M_3^{**}, therefore M_2' is less than M_3^{**}.

Thus we can say that M_1 has a greater ratio to M_3 than M_2 has to M_3.

On the other hand, it is also true that M_3'' is greater than M_2' and also that M_3^{**} is less than M_1'. Along with the fact that M_3^{**} is any multiple of M_3, and that M_1' and M_2' are equimultiples of M_1 and M_2, thus M_3 has a greater ratio to M_2 than M_3 has to M_1.

■

Modern Restatement

Let M_1 and M_2 be unequal magnitudes such that $M_1 \succ M_2$. Also let M_3 be a magnitude of any size. Then $M_1 : M_3 \succ M_2 : M_3$ and $M_3 : M_2 \succ M_3 : M_1$.

Modern Proof. Since $M_1 \succ M_2$ then $M_1 - M_2 \succ 0$.

Case I: $(M_1 - M_2) \prec M_2$

Choose k such that the multiple $k\,(M_1 - M_2)$ is greater than M_3.

$$k\,(M_1 - M_2) \succ M_3 \tag{3.8.1}$$

Then also $kM_2 \succ M_3$.

Now choose ℓ such that the multiple ℓM_3 is the largest multiple of M_3 that is less than or equal to kM_2. Thus $(\ell + 1)\,M_3$ will be greater than kM_2. In other words,

$$\ell M_3 \preceq kM_2 \prec (\ell + 1)\,M_3 \tag{3.8.2}$$

Also,

$$\begin{aligned} k\,(M_1 - M_2) + kM_2 &\cong kM_1 - kM_2 + kM_2 && (3.8.3)\\ &\cong kM_1 && (3.8.4) \end{aligned}$$

by Proposition 5. Thus,

$$\begin{aligned} k\left(M_1 - M_2\right) + kM_2 \succ & M_3 + \ell M_3 && (3.8.5) \\ kM_1 \succ & M_3 + \ell M_3 && (3.8.6) \\ kM_1 \succ & \left(\ell + 1\right) M_3 && (3.8.7) \end{aligned}$$

by Proposition 2.

At the same time,

$$kM_2 \prec \left(\ell + 1\right) M_3. \tag{3.8.8}$$

Therefore

$$M_1 : M_3 \succ M_2 : M_3$$

.

Also,

$$\left(\ell + 1\right) M_3 \succ kM_2 \tag{3.8.9}$$

and

$$\left(\ell + 1\right) M_3 \prec kM_1 \tag{3.8.10}$$

Therefore

$$M_3 : M_2 \succ M_3 : M_1$$

.

Case II: $M_2 \preceq M_1 - M_2$

Chose k such that

$$kM_2 \succ M_3. \tag{3.8.11}$$

Choose ℓ such that ℓM_3 is the largest multiple of M_3 that is less than or equal to $k\left(M_1 - M_2\right)$. Thus $\left(\ell + 1\right) M_3$, the very next multiple of M_3, will be greater than $k\left(M_1 - M_2\right)$.

$$\ell M_3 \preceq k\left(M_1 - M_2\right) \prec \left(\ell + 1\right) M_3 \tag{3.8.12}$$

Again,

$$\begin{aligned} k\left(M_1 - M_2\right) + kM_2 \cong & kM_1 - kM_2 + kM_2 && (3.8.13) \\ \cong & kM_1 && (3.8.14) \end{aligned}$$

by Proposition 5.

Now $kM_2 \succ M_3$, so

$$k(M_1 - M_2) + kM_2 \succ lM_3 + M_3 \quad (3.8.15)$$
$$kM_1 \succ (\ell + 1) M_3 \quad (3.8.16)$$

And since $kM_1 \succ (\ell + 1) M_3$ and $kM_2 \prec (\ell + 1) M_3$, thus

$$M_1 : M_3 \succ M_2 : M_3$$

.

As in Case I, by rewriting the inequalities we obtain

$$M_3 : M_2 \succ M_3 : M_1$$

.

■

3.9 Proposition IX

Euclid's Statement

Magnitudes which have the same ratio to the same are equal to one another; and magnitudes to which the same has the same ratio are equal.

Restatement

Let M_1 and M_2 have the same ratio to M_3. Then M_1 is equal to M_2. Also if M_3 has the same ratio to M_1 that M_3 has to M_2, then M_1 is equal to M_2.

Proof following Euclid. If M_1 is not equal to M_2, then M_1 would not have the same ratio to M_3 that M_2 has to M_3. Thus this is a contradiction and M_1 must equal M_2.

Similarly, if M_1 is not equal to M_2, then M_3 would not have the same ratio to M_1 that M_3 has to M_2. But it does, so this is also a contradiction and M_1 must equal M_2. ■

Modern Restatement

If $M_1 : M_3 \cong M_2 : M_3$ then $M_1 \cong M_2$, and if $M_3 : M_1 \cong M_3 : M_2$ then $M_1 \cong M_2$.

Modern Proof. If $M_1 \ncong M_2$, then either $M_1 \succ M_2$ or $M_1 \prec M_2$. So by Proposition VII, either $M_1 : M_3 \succ M_2 : M_3$ or $M_1 : M_3 \prec M_2 : M_3$. Which is a contradiction, thus $M_1 \cong M_2$.

Similarly, if $M_1 \ncong M_2$, then either $M_1 \succ M_2$ or $M_1 \prec M_2$. So by Proposition VII, either $M_3 : M_2 \succ M_3 : M_1$ or $M_3 : M_2 \prec M_3 : M_1$. Again we have a contradiction, thus $M_1 \cong M_2$.

■

3.10 Proposition X

Euclid's Statement

Of magnitudes which have a ratio to the same, that which has a greater ratio is greater; and that to which the same has a greater ratio is less.

Restatement

If M_1 has to M_3 a greater ratio than M_2 has to M_3, then M_1 is greater than M_2. Also if M_3 has to M_2 a greater ratio than M_3 has to M_1, then M_2 is less than M_1.

Proof following Euclid. First show that If M_1 has to M_3 a greater ratio than M_2 has to M_3, then M_1 is greater than M_2. If M_1 is not greater than M_2, then either M_1 is equal to M_2, or M_1 is less than M_2.

But M_1 is not equal to M_2 or else M_1 would have the same ratio to M_3 that M_2 has to M_3, and it does not. Also M_1 is not less than M_2, otherwise M_1 would have a lesser ratio to M_3 than M_2 has to M_3, and this is also not the case. Therefore M_1 is not equal to M_2, and M_1 is not less than M_2. Leaving the only possibility to be that M_1 is greater than M_2.

Next show that if M_3 has to M_2 a greater ratio than M_3 has to M_1, then M_2 is less than M_1. If M_2 is not less than M_1, then either M_2 is equal to M_1, or M_2 is greater than M_1.

But M_2 is not equal to M_1, otherwise M_3 would have to M_2 the same ratio that M_3 has to M_1, and it does not. Also, M_2 is not greater than M_1, because if it were M_3 would have to M_2 a lesser ratio than M_3 has to M_1, and this is not the case either. Therefore M_2 is not equal to M_1 and M_2 is not greater than M_1, which implies that M_2 is less than M_1. ■

Modern Restatement

If $M_1 : M_3 \succ M_2 : M_3$, then $M_1 \succ M_2$. Also if $M_3 : M_2 \succ M_3 : M_1$, then $M_2 \prec M_1$.

Modern Proof. Begin by showing that if $M_1 : M_3 \succ M_2 : M_3$, then $M_1 \succ M_2$. First assume that $M_1 \nsucc M_2$, then $M_1 \cong M_2$ or $M_1 \prec M_2$. But, by Proposition VII, if $M_1 \cong M_2$, then $M_1 : M_3 \cong M_2 : M_3$; and if $M_1 \prec M_2$, then by Proposition VIII $M_1 : M_3 \prec M_2 : M_3$. Thus $M_1 \cong M_2$ and $M_1 < M_2$ contradict $M_1 : M_3 \succ M_2 : M_3$. Therefore $M_1 \nprec M_2$ and $M_1 \ncong M_2$, which implies $M_1 \succ M_2$.

Similarly, if $M_3 : M_2 \succ M_3 : M_1$, then $M_2 \prec M_1$. Assume that $M_2 \not\prec M_1$, thus $M_2 \cong M_1$ or $M_2 \succ M_1$. Again by Proposition VII, if $M_2 \cong M_1$, then $M_3 : M_2 \cong M_3 : M_1$; also if $M_2 \succ M_1$, then by Proposition VIII $M_3 : M_2 \prec M_3 : M_1$. Thus $M_2 \ncong M_1$ and $M_2 \not\succ M_1$, leaving only $M_2 \prec M_1$ as a possibility. ■

3.11 Proposition XI

Euclid's Statement

Ratios which are the same with the same ratio are also the same with one another.

Restatement

If M_1 has the same ratio to M_2 that M_3 has to M_4, and M_3 has the same ratio to M_4 as M_5 has to M_6, then M_1 has the same ratio to M_2 that M_5 has to M_6.

Proof following Euclid. Let M_1', M_3', and M_5' be equimultiples of M_1, M_3, and M_5 respectively. Also let M_2^*, M_4^*, and M_6^* be equimultiples of M_2, M_4, and M_6.

Since M_1 has the same ratio to M_2 as M_3 has to M_4, if M_1' is in excess of M_2^*, then M_3' is in excess of M_4^*. If M_1' is in equal to M_2^*, then M_3' is equal to M_4^*. M_1' is less than M_2^*, then M_3' is less than M_4^*.

Similarly, since M_3 has the same ratio to M_4 as M_5 has to M_6, if M_3' is in excess of M_4^*, then M_5' is in excess of M_6^*. If M_3' is in equal to M_4^*, then M_5' is equal to M_6^*. M_3' is less than M_4^*, then M_5' is less than M_6^*.

So if M_1' is in excess of M_2^*, then M_3' is in excess of M_4^*, if equal, equal, and if less, less; but if M_3' is in excess of M_4*, then also M_5' is in excess of M_6^*, if equal, equal, if less, less. Thus if M_1' is in excess of M_2^*, then M_5' is in excess of M_6^*; if equal, equal; if less, less.

Therefore M_1 has to M_2 the same ratio that M_5 has to M_6. ■

Modern Restatement

If $M_1 : M_2 \cong M_3 : M_4$ and $M_3 : M_4 \cong M_5 : M_6$, then $M_1 : M_2 \cong M_5 : M_6$.

Modern Proof. Let kM_1, kM_3, and kM_5 be equimultiples of M_1, M_3, and M_5. Also let ℓM_2, ℓM_4, and ℓM_6 be equimultiples of M_2, M_4, and M_6.

Since $M_1 : M_2 \cong M : 3 : M_4$, if $kM_1 \succ \ell M_2$, then $kM_3 \succ \ell M_4$. If $kM_1 \cong \ell M_2$, then $kM_3 \cong \ell M_4$. $kM_1 \prec \ell M_2$, then $kM_3 \prec \ell M_4$.

Since $M_3 : M_4 \cong M : 5 : M_6$, if $kM_3 \succ \ell M_4$, then $kM_5 \succ \ell M_6$. If $kM_3 \cong \ell M_4$, then $kM_5 \cong \ell M_6$. $kM_3 \prec \ell M_4$, then $kM_5 \prec \ell M_6$.

Thus if $kM_1 \succ \ell M_4$, then $kM_5 \succ \ell M_6$. If $kM_1 \cong \ell M_2$, then $kM_5 \cong \ell M_6$. $kM_1 \prec \ell M_2$, then $kM_5 \prec \ell M_6$.

Therefore $M_1 : M_2 \cong M_5 : M_6$. ■

3.12 Proposition XII

Definition 3.12.1. Let magnitudes which have the same ratio be called *proportional.*

Euclid's Statement

If any number of magnitudes be proportional, as one of the antecedents is to one of the consequents, so will all the antecedents be to all the consequents.

Restatement

Let any number of magnitudes $M_1, \ldots M_n$ be proportional, so that M_1 is to M_2 as M_3 is to M_4, and as M_5 is to M_6, and so on. Then M_1 has the same ratio to M_2 that $M_1 + M_3 + M_5 + \cdots + M_{n-1}$ has to $M_2 + M_4 + M_6 + \cdots + M_n$.

Proof following to Euclid. Let $M_1, \ldots, M_6$ be proportional, so that M_1 is to M_2 as M_3 is to M_4 and M_5 is to M_6.

Take equimultiples M_1', M_3', and M_5' of M_1,M_3, and M_5 respectively. Also take any equimultiples of M_2^*, M_4^*, and M_6^* of M_2,M_4, and M_6 respectively.

Thus if M_1' is in excess of M_2^*, then M_3' is in excess of M_4^*, and M_5' is in excess of M_6^*. Also if M_1' is equal to M_2^*, then M_3' is equal to M_4^*, and M_5' is equal to M_6^*. And if if M_1' is less than M_2^*, then M_3' is less than M_4^*, and M_5' is less than M_6^*.

Thus in addition if M_1' is in excess of M_2^*, then $M_1' + M_3' + M_5'$ is in excess of $M_2^* + M_4^* + M_6^*$; if equal to, equal equal to; if less than, less than.

Now M_1' and $M_1' + M_3' + M_5'$ are equimultiples of M_1 and $M_1 + M_3 + M_5$, by Proposition 1.

Similarly M_2^* and $M_2^* + M_4^* + M_6^*$ are equimultiples of M_2 and $M_2 + M_4 + M_6$.

Therefore M_1 has the same ratio to M_2 as $M_1 + M_3 + M_5$ has to $M_2 + M_4 + M_6$, by Euclid's Definition V.

Therefore if any number of magnitudes are proportional then, as one of the antecedents is to one of the consequents, so will all the antecedents be to all of the consequents. ■

Modern Restatement

If $M_1 : M_2 \cong M_3 : M_4 \cong \ldots \cong M_{n-1} : M_n$, then $M_1 : M_2 \cong M_1 + M_3 + \cdots + M_{n-1} : M_2 + M_4 + \cdots + M_n$, for $n \in \mathbb{N}$, such that n is a multiple of 2.

Modern Proof. Let $\Theta(n)$ be the statement:

$$\text{If } M_1 : M_2 \cong M_3 : M_4 \cong \ldots \cong M_{n-1} : M_n, \text{ then}$$
$$M_1 : M_2 \cong M_1 + M_3 + \cdots + M_{n-1} : M_2 + M_4 + \cdots + M_n$$

Claim: $\forall n$, $\Theta(n)$.
1) $\Theta(2)$ is obvious.
2) $\Theta(4)$.
In other words, if $M_1 : M_2 \cong M_3 : M_4$ then, $M_1 : M_2 \cong M_1 + M_3 : M_2 + M_4$.
Since $M_1 : M_2 \cong M_3 : M_4$, thus if $kM_1 \succ \ell M_2$, then $kM_2 \succ \ell M_4$; if $kM_1 \cong \ell M_2$, then $kM_2 \cong \ell M_4$, $kM_1 \prec \ell M_2$, then $kM_2 \prec \ell M_4$.
But if $kM_1 \succ \ell M_2$ and $kM_3 \succ \ell M_4$, then

$$\begin{aligned} kM_1 + kM_3 &\succ \ell M_1 + \ell M_4, && \text{by Addition inequality,} \\ k(M_1 + M_3) &\succ \ell(M_1 + M_3), && \text{Proposition 1.} \end{aligned}$$

Similarly, if $kM_1 \cong \ell M_2$, then

$$\begin{aligned} kM_1 + kM_3 &\cong \ell M_1 + \ell M_4, && \text{by Addition inequality,} \\ k(M_1 + M_3) &\cong \ell(M_1 + M_3), && \text{Proposition 1.} \end{aligned}$$

and

$$\begin{aligned} kM_1 + kM_3 &\prec \ell M_1 + \ell M_4, && \text{by Addition inequality,} \\ k(M_1 + M_3) &\prec \ell(M_1 + M_3), && \text{by Proposition 1.} \end{aligned}$$

Therefore $M_1 : M_2 \cong M_1 + M_3 : M_2 + M_4$
3)$\Theta(n-2) \Rightarrow \Theta(n)$.

$$\begin{aligned} M_1 : M_2 &\cong M_1 + \cdots + M_{(n-2)-1} : M_2 + \cdots + M_{n-2}\ , && \Theta(n-2), \\ &\cong \left(M_1 + \cdots + M_{(n-2)-1}\right) + M_{n-1} : (M_2 + \cdots + M_{n-2}) + M_n\ , && \text{by } \Theta(4), \\ &\cong M_1 + \cdots + M_{(n-2)-1} + M_{n-1} : M_2 + \cdots + M_{n-2} + M_n, && \text{associative property.} \end{aligned}$$

■

3.13 Proposition XIII

Euclid's Statement

If a first magnitude have to a second the same ratio as a third to a fourth, and the third have to the fourth a greater ratio than a fifth has to a sixth, the first will also have to a second a greater ratio tan the fifth to the sixth.

Restatement

Let M_1 have to M_2 the same ratio that M_3 has to M_4, and let M_3 have to M_4 a greater ratio than M_5 has to M_6. Then $M_1 : M_2$ has a greater ratio than M_5 has to M_6.

Proof following to Euclid. Since M_3 has to M_4 a greater ratio than M_5 has to M_6, then let M_3' and M_5' be equimultiples of M_3 and M_5 and let M_4^* and M_6^* be equimultiples of M_4 and M_6, such that M_3' exceeds M_4^*, while M_5' does not exceed M_6^*. [Definition 7]

Let M_1' be the same multiple of M_1 that M_3' is of M_3, and let M_2^* be the same multiple of M_2 that M_4^* is of M_4.

Since M_1 has the same ratio to M_2 that M_3 has to M_4, and equimultiples M_1' and M_3' of M_1 and M_3 have been taken, and equimultiples M_2^* and M_4^* of M_2 and M_4 have also been taken; then if M_1' is greater than M_2^* so M_3' is also greater than M_4^*; if equal, equal; if less, less.

But since M_3' exceeds M_4^*, then M_1' must also exceed M_2^*, because M_1 has to M_2 the same ratio that M_3 has to M_4. At the same time M_5' does not exceed M_6^*, while M_1' and M_5' are equimultiples of M_1 and M_5, and M_2^* and M_6^* are equimultiples of M_2 and M_6. Therefore M_1 has to M_2 a greater ratio than M_5 has to M_6.

■

Modern Restatement

If $M_1 : M_2 \cong M_3 : M_4$ and $M_3 : M_4 \succ M_5 : M_6$, then $M_1 : M_2 \succ M_5 : M_6$.

Modern Proof. Since $M_3 : M_4 \succ M_5 : M_6$, let kM_3 and kM_5 be equimultiples of M_3 and M_5, while ℓM_4 and ℓM_6 are equimultiples of M_4 and M_6, such that $kM_3 \succ \ell M_4$ and $kM_5 \preceq \ell M_6$. [Def. 7]

Also let kM_1 be a multiple of M_1, and ℓM_2 a multiple of M_2.

Since $M_1 : M_2 \cong M_3 : M_4$, then if equimultiples kM_1 and kM_3 of M_1 and M_3, and equimultiples ℓM_2 and ℓM_4 of M_2 and M_4 have been taken, then if

$kM_1 \succ \ell M_2 \Leftrightarrow kM_3 \succ \ell M_4$, and if

$kM_1 \cong \ell M_2 \Leftrightarrow kM_3 \cong \ell M_4$, and if

$kM_1 \prec \ell M_2 \Leftrightarrow kM_3 \prec \ell M_4$.

But $kM_3 \succ \ell M_4$ thus $kM_1 \succ \ell M_2$.

Thus $kM_1 \succ \ell M_2$ and $kM_5 \preceq \ell M_6$, therefore $M_1 : M_2 \succ M_5 : M_6$.

■

3.14 Proposition XIV

Euclid's Statement

If a first magnitude have to a second the same ratio as a third to a fourth, and the first be greater than the third, the second will also be greater than the fourth; if equal, equal; and if less, less.

Restatement

Suppose M_1 has to M_2 the same ratio as M_3 has to M_4. Also suppose that M_1 is greater than M_3. Then M_2 is greater than M_4. If M_1 is equal to M_3, then M_2 is equal to M_4. If M_1 is less than M_3, then M_2 is less than M_4.

Proof following Euclid. Since M_1 exceeds M_3, then M_1 has to M_2 a greater ratio than M_3 has to M_2 by Proposition VIII. But M_1 has to M_2 the same ratio as M_3 has to M_4, thus M_3 has to M_4 a greater ratio than M_3 has to M_2 by Proposition XIII. Therefore, by Proposition X, M_4 is less than M_2, in other words, M_2 is greater than M_4.

If M_1 is equal to M_3, then M_1 has to M_2 the same ratio as M_3 has to M_2. For if M_1 is equal to M_3 then equimultiples M_1' and M_3' of M_1 and M_3 are also equal. Then for any multiple M_2^* of M_2, if M_1' exceeds M_2^*, then M_3' also exceeds M_2^*; if equal, equal; if less, less. But M_1 has to M_2 the same ratio as M_3 has to M_4, thus M_3 has to M_2 the same ratio as M_3 has to M_4. Therefore, by Proposition IX, M_3 is equal to M_4.

Similarly, since M_1 is less than M_3, then M_1 has to M_2 a smaller ratio than M_3 has to M_2 by Proposition VIII. But M_1 has to M_2 the same ratio as M_3 has to M_4, thus M_3 has to M_4 a smaller ratio than M_3 has to M_2 by Proposition XIII. Therefore, by Proposition X, M_2 is less than M_4. ■

Modern Restatement

Suppose $M_1 : M_2 \cong M_3 : M_4$. Then $M_1 \succeq M_3 \Leftrightarrow M_2 \succeq M_4$.

Modern Proof.

$$\begin{aligned} M_1 &\succ M_3, && \text{by assumption,} \\ kM_1 &\succ kM_3, && \text{Property of equimultiples.} \end{aligned}$$

Thus for any multiple ℓM_2 of M_2,

$$\begin{aligned} M_1 : M_2 &\succ M_3 : M_2, && \text{by Proposition VIII,} \\ M_1 : M_2 &\cong M_3 : M_4, && \text{hypothesis statement,} \\ M_3 : M_4 &\succ M_3 : M_2, && \text{by Proposition XIII,} \\ M_2 &\succ M_4, && \text{by Proposition X.} \end{aligned}$$

Now suppose $M_1 \cong M_3$

$$\begin{aligned} M_1 &\cong M_3, && \text{by assumption,} \\ M_1 : M_2 &\cong M_3 : M_2, && \text{Corollary to Proposition IX.} \end{aligned}$$

Since $M_1 : M_2 \cong M_3 : M_2$ and $M_1 : M_2 \cong M_3 : M_2$, then $M_3 : M_2 \cong M_3 : M_4$. Therefore, by Proposition IX, $M_2 \cong M_4$.

Similarly, if $M_1 \prec M_3$ then $M_2 \prec M_4$

$$
\begin{aligned}
M_1 &\prec M_3, && \text{by assumption,}\\
kM_1 &\prec kM_3, && \text{Property of equimultiples.}
\end{aligned}
$$

Thus for any multiple ℓM_4 of M_4,

$$
\begin{aligned}
M_1 : M_4 &\prec M_3 : M_4, && \text{by Proposition VIII,}\\
M_1 : M_2 &\cong M_3 : M_4, && \text{by the hypothesis statement,}\\
M_1 : M_4 &\prec M_1 : M_2, && \text{by Proposition XIII,}\\
M_2 &\prec M_4, && \text{by Proposition X.}
\end{aligned}
$$

■

3.15 Proposition XV

Euclid's Statement

Parts have the same ratio as the multiple of them taken in corresponding order.

Restatement

Suppose M_1' and M_2' are equimultiples of M_1 and M_2, then M_1 has to M_2 the same ratio that M_1' has to M_2'.

Proof following Euclid. Since M_1' and M_2' are equimultiples of M_1 and M_2, then as many parts of M_1' that are equal to M_1, is the same as the number of parts of M_2' that are equal to M_2.

When M_1' and M_2' are equimultiples of M_1 and M_2, it is possible to divide M_1' into parts equal to M_1 without remainder, and divide M_2' into parts equal to M_2 without remainder.

Also when each part of M_1' is equal to M_1, thus they are equal to each other. Similarly, since each part of M_2' is equal to M_2 then they are also equal to each other.

Since every part of M_1' is equal to every other part of M_1', and since every part of M_2' is equal to every other part of M_2', then the ratio of the any part of M_1' to any part of M_2' is equal to the ratio of another part of M_1' to another part of M_2' by proposition VII.

Thus as one part of M_1' is to a part of M_2' the sums of parts M_1' is to the sum M_2', by Proposition XII.

Therefore M_1 to M_2 has the same ratio as M_1' is to M_2'. ■

Modern Restatement

Let kM_1 and kM_2 be equimultiples of M_1 and M_2, then $M_1 : M_2 \cong kM_1 : kM_2$.

Modern Proof. Let $\Theta(k)$ be the statement: $M_1 : M_2 \cong kM_1 : kM_2$, then $\Theta(k) \quad \forall k \in \mathbb{N}$.

We show by induction that $M_1 : M_2 \cong kM_1 : kM_2$.

1) Claim: $\Theta(1)$, in other words $M_1 : M_2 \cong 1M_1 : 1M_2$. Follows directly from the definition of multiple.

2) Claim $\Theta(k-1) \Rightarrow \Theta(k) \quad \forall k \in \mathbb{N}$, such that $k > 1$.

Follows from Proposition XII, letting $M_3 \cong M_1$, $M_4 \cong M_2$, $\ldots$, $M_{2k+1} \cong M_1$, $M_{2k+2} \cong M_2$.

■

3.16 Proposition XVI

Euclid's Statement

If four magnitudes be proportional, they will also be proportional alternately.

Restatement

If M_1 have the same ratio to M_2 that M_3 has to M_4, then M_1 will also have the same ratio to M_3 that M_2 has to M_4.

Proof following to Euclid. Take equimultiples $M_1\prime$ and M_2' of M_1 and M_2, and equimultiples M_3^* and M_4^* of M_3 and M_4.

By Proposition XV, M_1 is to M_2 the same ratio as M_1' is to M_2', and M_3 is to M_4 the same ratio as M_3^* is to M_4^*.

Since M_1 is to M_2 as M_3 is to M_4, and M_1 is to M_2 the same ratio as M_1' is to M_2', then by Proposition XI M_3 has to M_4 the same ratio that M_1' has to M_2'. Also M_3 is to M_4 the same ratio as M_3^* is to M_4^*, thus M_1' is to M_2' the same ratio as M_3^* is to M_4^*.

Now since M_1' is to M_2' the same ratio as M_3^* is to M_4^*, then by Proposition XIV, if M_1' is exceeds M_3^* then M_2' exceeds M_4^*, if M_1' is equal to M_3^* then M_2' is equal to M_4^*, and if M_1' is falls short of M_3^* then M_2' falls short of M_4^*.

Also since $M_1\prime$ and M_2' are equimultiples of M_1 and M_2, and M_3^* and M_4^* are equimultiples of M_3 and M_4, then M_1 will have the same ratio to M_3 that M_2 has to M_4.

■

Modern Restatement

If $M_1 : M_2 \cong M_3 : M_4$, then $M_1 : M_3 \cong M_2 : M_4$.

Modern Proof. Since $M_1 : M_2 \cong M_3 : M_4$, and

$$
\begin{aligned}
M_1 : M_2 &\cong kM_1 : kM_2, && \text{by Proposition XV,}\\
M_3 : M_4 &\cong kM_1 : kM_2, && \text{by Proposition XI,}\\
M_3 : M_4 &\cong \ell M_3 : \ell M_4, && \text{by Proposition XV,}\\
kM_1 : kM_2 &\cong \ell M_3 : \ell M_4, && \text{by Proposition XI.}
\end{aligned}
$$

Then by Proposition XIV, since $kM_1 : kM_2 \cong \ell M_3 : \ell M_4$, then

$$kM_1 \succeq \ell M_3 \qquad \Leftrightarrow \qquad kM_2 \succeq \ell M_4.$$

Therefore $M_1 : M_3 \cong M_2 : M_4$.

■

3.17 Proposition XVII

Euclid's Statement

If magnitudes be proportional componendo, they will also be proportional separando.

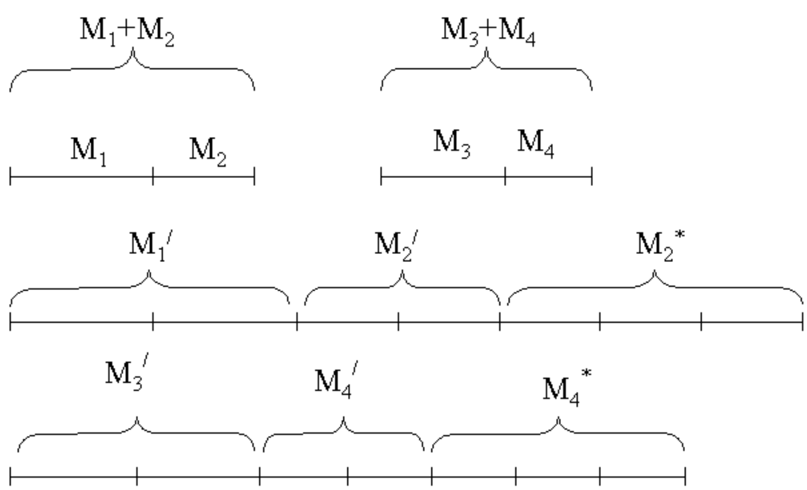

Figure 3.7: Proposition 17

Restatement

If $M_1, \ldots, M_4$ be magnitudes and $(M_1 + M_2)$ is to M_2 the same ratio as $(M_3 + M_4)$ is to M_4, then M_1 has the same ratio to M_2 that M_3 has to M_4.

Proof following to Euclid. Let M_1', M_2', M_3', and M_4' be equimultiples of M_1, M_2, M_3, and M_4. Also let M_2^* and M_4^* be equimultiples of M_2 and M_4.

Since M_1' is the same multiple of M_1 that M_2' is of M_2, then by Proposition I M_1' is the same multiple of M_1 that $M_1' + M_2'$ is of $M_1 + M_2$.

Also since M_1' is the same multiple of M_1 that M_3' is of M_3, then $M_1' + M_2'$ is the same multiple of $M_1 + M_2$ that M_3' is of M_3.

Since M_3' is the same multiple of M_3 that M_4' is of M_4, then by Proposition I M_3' is the same multiple of M_3 that $M_3' + M_4'$ is of $M_3 + M_4$. Thus $M_1' + M_2'$ is the same multiple of $M_1 + M_2$ that $M_3' + M_4'$ is of $M_3 + M_4$.

Recall that M_2' and M_4' are equimultiples of M_2 and M_4, and since M_2^* and M_4^* are equimultiples of M_2 and M_4, then $M_2' + M_2^*$ and $M_4' + M_4^*$ are equimultiples of M_2 and M_4 by Proposition II.

Since $(M_1 + M_2)$ is to M_2 the same ratio as $(M_3 + M_4)$ is to M_4, and equimultiples $M_1' + M_2'$ and $M_3' + M_4'$ have been taken of $M_1 + M_2$ and $M_3 + M_4$, and equimultiples $M_2' + M_2^*$ and $M_4' + M_4^*$ of M_2 and M_4 have also been taken; therefore if $M_1' + M_2'$ is in excess of $M_2' + M_2^*$, then $M_3' + M_4'$ is in excess of $M_4' + M_4^*$; if equal, equal; if less, less.

Suppose $M_1' + M_2'$ exceeds and $M_2' + M_2^*$, then by subtracting M_2' from each magnitude M_1' exceeds M_2^*. But if $M_1' + M_2'$ exceeds $M_2' + M_2^*$, then $M_3' + M_4'$ exceeds $M_4' + M_4^*$, and by subtracting the magnitude M_4' from each, M_3' will exceed M_4^*.

Again, suppose $M_1' + M_2'$ is equal to $M_2' + M_2^*$, then by subtracting M_2' from each magnitude M_1' is equal to M_2^*. But if $M_1' + M_2'$ is equal to $M_2' + M_2^*$, then $M_3' + M_4'$ is also equal to $M_4' + M_4^*$, and by subtracting the magnitude M_4' from each, M_3' is equal to M_4^*.

Similarly, suppose $M_1' + M_2'$ is less than $M_2' + M_2^*$, then by subtracting M_2' from each magnitude M_1' is less than M_2^*. But if $M_1' + M_2'$ is less than $M_2' + M_2^*$, then $M_3' + M_4'$ is also less than $M_4' + M_4^*$, and by subtracting the magnitude M_4' from each, M_3' will be less than M_4^*.

Since M_1' and M_3' are equimultiples of M_1 and M_3, and M_2^* and M_4^* are equimultiples of M_2 and M_4, therefore M_1 has to M_2 the same ratio that M_3 has to M_4.

■

Modern Restatement

If $M_1, \ldots, M_4$ be magnitudes and $(M_1 + M_2) : M_2 \cong (M_3 + M_4) : M_4$, then $M_1 : M_2 \cong M_3 : M_4$.

Modern Proof. Take equimultiples kM_1, kM_2, kM_3, and kM_4 of M_1, M_2, M_3, and M_4. Also take equimultiples ℓM_2 and ℓM_4 of M_2 and M_4.

$$
\begin{aligned}
k(M_1+M_2) &\cong kM_1+kM_2 \quad \text{and} \quad k(M_3+M_4)\cong kM_3+kM_4 \quad \text{by Proposition I,}\\
kM_2+\ell M_2 &\cong (k+\ell)M_2 \quad \text{and} \quad kM_4+\ell M_4 \cong (k+\ell)M_4 \quad \text{by Proposition II.}
\end{aligned}
$$

Since $(M_1+M_2) : M_2 \cong (M_3+M_4) : M_4$, and kM_1+kM_2 is the same multiple of (M_1+M_2) that $kM_3+kM_4 \succ kM_4$ is of (M_3+M_4), and $kM_2+\ell M_2$ is the same multiple of M_2 that $kM_4+\ell M_4$ is of M_4, then

$$
\begin{array}{llll}
\text{if} & kM_1+kM_2 \succ kM_2+\ell M_2, & \text{then} & kM_3+kM_4 \succ kM_4+\ell M_4,\\
\text{if} & kM_1+kM_2 \cong kM_2+\ell M_2, & \text{then} & kM_3+kM_4 \cong kM_4+\ell M_4,\\
\text{if} & kM_1+kM_2 \prec kM_2+\ell M_2, & \text{then} & kM_3+kM_4 \prec kM_4+\ell M_4.
\end{array}
$$

By subtracting kM_2 from each side of every inequalities on the left hand side, and subtracting kM_4 from the inequalities on the right hand side we are left with

$$
\begin{array}{lcl}
kM_1 \succ \ell M_2, & \Leftrightarrow & kM_3 \succ \ell M_4,\\
kM_1 \cong \ell M_2, & \Leftrightarrow & kM_3 \cong \ell M_4,\\
kM_1 \prec \ell M_2, & \Leftrightarrow & kM_3 \prec \ell M_4.
\end{array}
$$

Therefore $M_1 : M_2 \cong M_3 : M_4$.

■

3.18 Proposition XVIII

Euclid's Statement

If magnitudes be proportional separando, they will also be proportional componendo.

Restatement

If $M_1, \ldots, M_4$ be magnitudes and $M_1 : M_2 \cong M_3 : M_4$, then $(M_1+M_2) : M_2 \cong (M_3+M_4) : M_4$.

Proof following to Euclid. If M_1+M_2 has to M_2 a different ratio to M_3+M_4 has to M_4, then M_1+M_2 has to M_2 the same ratio as M_3+M_4 has to a magnitude M^* less than or greater than M_4.

Suppose the magnitude M^* is less than M_4.

Since M_1+M_2 is to M_2 as M_3+M_4 is to M^*, then by Proposition XVII M_1 is to M_2 the same ratio as $M_3+M_4-M^*$ is to M^*.

Recall that M_1 is to M_2 as M_3 is to M_4, thus by Proposition XI $M_3 + M_4 - M^*$ is to M^* as M_3 is to M_4.

Since M^* is less than M_4, then $M_3+M_4-M^*$ is greater than M_3. By Proposition XIV, since $M_3 + M_4 - M^*$ is greater than M_3, then M_* is greater than M_4, but we have a contradiction since M_* was chosen to be less than M_4.

Similarly we have a contradiction if we chose M^* to be greater than M_4. Since $M_1 + M_2$ is to M_2 as $M_3 + M_4$ is to M^*, and by Proposition XVII M_1 is to M_2 the same ratio as $M_3 + M_4 - M^*$ is to M^*.

Recall that M_1 is to M_2 as M_3 is to M_4, thus by Proposition XI $M_3 + M_4 - M^*$ is to M^* as M_3 is to M_4.

Since M^* is greater than M_4, then $M_3+M_4-M^*$ is less than M_3. By Proposition XIV, since $M_3 + M_4 - M^*$ is less than M_3, then M_* is less than M_4, but we have a contradiction since M_* was chosen to be greater than M_4.

Since M^* is not less than M_4 and M^* is not greater than M_4, then M^* is equal to M_4. Therefore $M_1 + M_2$ is to M_2 as $M_3 + M_4$ is to M_4.

■

Modern Restatement

If $M_1 : M_2 \cong M_3 : M_4$, then $(M_1 + M_2) : M_2 \cong (M_3 + M_4) : M_4$.

Modern Proof.
Suppose $M_1 : M_2 \cong M_3 : M_4$.

Suppose that $(M_1 + M_2) : M_2 \ \cong \ (M_3 + M_4) : M^*$ where $M^* \prec M_4$.

Then $M_1 : M_2 \ \cong \ (M_3 + M_4 - M^*) : M^*$ by Proposition XVII.

But $M_3 + M_4 - M^* \succ M_3$.

So $(M_3 + M_4 - M^*) : M^* \succ M_3 : M_4$. Which is a contradiction.

Now Suppose that $(M_1 + M_2) : M_2 \ \cong \ (M_3 + M_4) : M^*$ where $M^* \succ M_4$.

Then $M_1 : M_2 \ \cong \ (M_3 + M_4 - M^*) : M^*$ by Proposition XVII.

But $M_3 + M_4 - M^* \prec M_3$.

So $(M_3 + M_4 - M^*) : M_* \prec M_3 : M_4$. Which is a contradiction.

Thus $(M_1 + M_2) : M_2 \ncong (M_3 + M_4) : M^*$ where $M^* \prec M_4$. and $(M_1 + M_2) : M_2 \ncong (M_3 + M_4) : M^*$ where $M^* \succ M_4$. Therefore $(M_1 + M_2) : M_2 \cong (M_3 + M_4) : M_4$.

■

3.19 Proposition XIX

Euclid's Statement

If, as a whole is to a whole, so is a part subtracted to a part subtracted, the remainder will also be to the remainder as whole to whole.

Restatement

If M_1 is to M_2 the same ratio as M_3 which is a part of M_1 is to M_4 which is a part of M_2, then $M_1 - M_3$ is to $M_2 - M_4$ the same ratio as M_1 is to M_2.

Proof following to Euclid. As M_1 is to M_2 the same ratio as M_3 is to M_4, then M_1 is to M_3 the same ratio as M_2 is to M_4, by Proposition XVI.

By Proposition XVII, since M_1 is the whole of M_3 and $(M_1 - M_3)$, and M_2 is the whole of M_4 and $(M_2 - M_4)$, then $(M_1 - M_3)$ is to M_3 the same ratio as $(M_2 - M_4)$ is to M_4.

Thus $(M_1 - M_3)$ is to $(M_2 - M_4)$ the same ratio as M_3 is to M_4. But M_1 is to M_2 the same ratio as M_3 is to M_4. So by Proposition XI, $(M_1 - M_3)$ is to $(M_2 - M_4)$ the same ratio as M_1 is to M_2. ■

Modern Restatement

If $M_1 : M_2 \cong M_3 : M_4$, such that $M_3 \prec M_1$ and $M_4 \prec M_2$, then $(M_1 - M_3) : (M_2 - M_4) \cong M_1 : M_2$.

Modern Proof.

$$\begin{aligned} M_1 : M_2 &\cong M_3 : M_4 && \text{by hypothesis,} \\ M_1 : M_3 &\cong M_2 : M_4 && \text{by Proposition XVI,} \end{aligned}$$

Since $M_3 \prec M_1$ and $M_4 \prec M_2$,

$$\begin{aligned} (M_1 - M_3) : M_3 &\cong (M_2 - M_4) : M_4 && \text{by Proposition XVII,} \\ (M_1 - M_3) : (M_2 - M_4) &\cong M_3 : M_4 && \text{by Proposition XVI,} \\ (M_1 - M_3) : (M_2 - M_4) &\cong M_1 : M_2 && \text{by Proposition XI.} \end{aligned}$$

■

Porism

Corollary 3.19.1. *Suppose that If $M_1 : M_3 \cong M_2 : M_4$, such that $M_3 \prec M_1$ and $M_4 \prec M_2$, then $M_1 : (M_1 - M_3) \cong M_2 : (M_2 - M_4)$*

Proof.

$$\begin{aligned} (M_1 - M_3) : M_3 &\cong (M_2 - M_4) : M_4 && \text{by Proposition XVII,} \\ M_3 : (M_1 - M_3) &\cong M_4 : (M_2 - M_4) && \text{by Corollary 3.7.1.} \end{aligned}$$

■

3.20 Proposition XX

Euclid's Statement

If there be three magnitudes, and others equal to them in multitude, which taken two and two are in the same ratio, and if ex aequali the first be greater than the third, the fourth will also be greater than the sixth; if equal, equal; and if less, less.

Restatement

Suppose M_1 is to M_2 the same ratio as M_4 is to M_5, and M_2 is to M_3 the same ratio as M_5 is to M_6. Thus if M_1 is greater than M_3, then M_4 is greater than M_6; if equal, equal; if less, less.

Proof following to Euclid. Since M_1 exceeds M_3, then by Proposition VIII M_1 has to M_2 a greater ratio than M_3 has to M_2. Also M_3 is to M_2 as M_6 is to M_5 by Corollary 3.7.1. But M_1 is to M_2 as M_4 is to M_5. By Proposition XIII M_4 is to M_5 a greater ratio than M_6 is to M_5, and M_4 is greater than M_6 by Proposition X.

Similarly if M_1 is less than M_3, then M_1 has to M_2 a lesser ratio than M_3 has to M_2. By Corollary 3.7.1 M_3 is to M_2 the same ratio as M_6 is to M_5. Also M_1 is to M_2 the same ratio that M_4 is to M_5. Therefore by Proposition XIII M_4 is to M_5 a lesser ratio than M_6 is to M_5, and by Proposition X M_4 is less than M_6.

If M_1 is equal to M_3, then by Proposition VII M_1 has to M_2 the same ratio as M_3 has to M_2. Again M_3 is to M_2 as M_6 is to M_5 by Corollary 3.7.1, and recall that M_1 is to M_2 the same ratio that M_4 is to M_5. Thus by Proposition XI, M_4 is to M_5 the same ratio that M_6 is to M_5. Therefore M_4 is equal to M_6 by Proposition IX. ■

Modern Restatement

Suppose $M_1 : M_2 \cong M_4 : M_5$, and $M_2 : M_3 \cong M_5 : M_6$. Thus if $M_1 \succ M_3$, then $M_4 \succ M_6$; if $M_1 \cong M_3$, then $M_4 \cong M_6$; if $M_1 \prec M_3$, then $M_4 \prec M_6$.

Modern Proof. If

$$
\begin{aligned}
M_1 &\succ M_3 && \text{by hypothesis,}\\
M_1 : M_2 &\succ M_3 : M_2 && \text{by Proposition VIII,}\\
M_1 : M_2 &\cong M_4 : M_5 && \text{by hypothesis,}\\
M_3 : M_2 &\cong M_6 : M_5 && \text{by Corollary 3.7.1,}\\
M_4 : M_5 &\succ M_6 : M_5 && \text{by Proposition XIII,}\\
M_4 &\succ M_6 && \text{by Proposition X.}
\end{aligned}
$$

Also if

$$\begin{aligned}
M_1 &\prec M_3 && \text{by hypothesis,}\\
M_1 : M_2 &\prec M_3 : M_2 && \text{by Proposition VIII,}\\
M_1 : M_2 &\cong M_4 : M_5 && \text{by hypothesis,}\\
M_3 : M_2 &\cong M_6 : M_5 && \text{by Corollary 3.7.1,}\\
M_4 : M_5 &\prec M_6 : M_5 && \text{by Proposition XIII,}\\
M_4 &\prec M_6 && \text{by Proposition X.}
\end{aligned}$$

Finally if

$$\begin{aligned}
M_1 &\cong M_3 && \text{by hypothesis,}\\
M_1 : M_2 &\cong M_3 : M_2 && \text{by Proposition VII,}\\
M_1 : M_2 &\cong M_4 : M_5 && \text{by hypothesis,}\\
M_3 : M_2 &\cong M_6 : M_5 && \text{by Corollary 3.7.1,}\\
M_4 : M_5 &\cong M_6 : M_5 && \text{by Proposition XI,}\\
M_4 &\cong M_6 && \text{by Proposition IX.}
\end{aligned}$$

Therefore when $M_1 : M_2 \cong M_4 : M_5$, and $M_2 : M_3 \cong M_5 : M_6$; if $M_1 \succ M_3$, then $M_4 \succ M_6$; if $M_1 \cong M_3$, then $M_4 \cong M_6$; if $M_1 \prec M_3$, then $M_4 \prec M_6$. ■

3.21 Proposition XXI

Euclid's Statement

If there be three magnitudes, and others equal to them in multitude, which taken two and two together are in the same ratio, and the proportion of them be perturbed, then, if ex aequali the first magnitude is greater than the third, the fourth will also be greater than the sixth; if equal, equal; if less, less.

Restatement

Suppose M_1 is to M_2 the same ratio as M_5 is to M_6, and M_2 is to M_3 the same ratio as M_4 is to M_5. Thus if M_1 is greater than M_3, then M_4 is greater than M_6; if equal, equal; if less, less.

Proof following to Euclid. Assume M_1 is greater than M_3, then M_1 is to M_2 a greater ratio than M_3 is to M_2 by Proposition VIII. But M_1 is to M_2 the same ratio as M_5 is to M_6. Also M_3 is to M_2 the same ratio as M_5 is to M_4. Thus, by Proposition XIII M_5 is to M_6 a greater ratio than M_5 has to M_4, and M_6 is less than M_4 by Proposition X. In other words M_4 is greater than M_6.

If M_1 is equal to M_3, then M_1 is to M_2 the same ratio as M_3 is to M_2 by Proposition VII. Since M_1 is to M_2 the same ratio as M_5 is to M_6, and M_3 is to

M_2 as M_5 is to M_4. Therefore M_5 is to M_6 the same ratio as M_5 is to M_4 by Proposition XI. Therefore M_4 is equal to M_6.

Similarly if M_1 is less than M_3, then M_1 is to M_2 a lesser ratio than M_3 is to M_2 by Proposition VIII. Since M_1 is to M_2 as M_5 is to M_6, and M_3 is to M_2 the same ratio as M_5 is to M_4, then by Proposition XIII, M_5 is to M_6 a lesser ratio than M_5 has to M_4, and M_4 is less than M_6.

■

Modern Restatement

Suppose $M_1 : M_2 \cong M_5 : M_6$, and $M_2 : M_3 \cong M_4 : M_5$. Thus if $M_1 \succ M_3$, then $M_4 \succ M_6$; if $M_1 \cong M_3$, then $M_4 \cong M_6$; if $M_1 \prec M_3$, then $M_4 \prec M_6$.

Modern Proof. If

$$\begin{aligned}
M_1 &\succ M_3 && \text{by hypothesis,}\\
M_1 : M_2 &\succ M_3 : M_2 && \text{by Proposition VIII,}\\
M_1 : M_2 &\cong M_5 : M_6 && \text{by hypothesis,}\\
M_3 : M_2 &\cong M_5 : M_4 && \text{by Corollary 3.7.1,}\\
M_5 : M_6 &\succ M_5 : M_4 && \text{by Proposition XIII,}\\
M_4 &> M_6 && \text{by Proposition X.}
\end{aligned}$$

Also if

$$\begin{aligned}
M_1 &\prec M_3 && \text{by hypothesis,}\\
M_1 : M_2 &\prec M_3 : M_2 && \text{by Proposition VIII,}\\
M_1 : M_2 &\cong M_5 : M_6 && \text{by hypothesis,}\\
M_3 : M_2 &\cong M_5 : M_4 && \text{by Corollary 3.7.1,}\\
M_4 : M_5 &\prec M_5 : M_4 && \text{by Proposition XIII,}\\
M_4 &\prec M_6 && \text{by Proposition X.}
\end{aligned}$$

Finally if

$$\begin{aligned}
M_1 &\cong M_3 && \text{by hypothesis,}\\
M_1 : M_2 &\cong M_3 : M_2 && \text{by Proposition VII,}\\
M_1 : M_2 &\cong M_5 : M_6 && \text{by hypothesis,}\\
M_3 : M_2 &\cong M_5 : M_4 && \text{by Corollary 3.7.1,}\\
M_5 : M_6 &\cong M_5 : M_4 && \text{by Proposition XI,}\\
M_4 &\cong M_6 && \text{by Proposition IX.}
\end{aligned}$$

Therefore when $M_1 : M_2 \cong M_5 : M_6$, and $M_2 : M_3 \cong M_4 : M_5$; if $M_1 \succ M_3$, then $M_4 \succ M_6$; if $M_1 \cong M_3$, then $M_4 \cong M_6$; if $M_1 \prec M_3$, then $M_4 \prec M_6$.

■

3.22 Proposition XXII

Euclid's Statement

If there be any number of magnitudes whatever, and others equal to them in multitude, which taken two and two together are in the same ratio, they will also be in the same ratio ex aequali.

Restatement

Suppose M_1, M_2, …, M_6 be magnitudes such that M_1 is to M_2 the same ratio as M_4 is to M_5, and M_2 is to M_3 the same ratio as M_5 is to M_6, then M_1 will have to M_3 the same ratio as M_4 to M_6.

Proof following to Euclid. Take equimtultiples M_1' and M_4' of M_1 and M_4, any equimultiples M_2^* and M_5^* of M_2 and M_5, and any equimultiles of $M_3^\star$ and $M_6^\star$ of M_3 and M_6.

Since M_1 is to M_2 as M_4 is to M_5, then by Proposition IV M_1' is to M_2^* as M_4' is to M_5^*. Also since M_2 is to M_3 as M_5 is to M_6, then M_2^* is to $M_3^\star$ as M_5^* is to $M_6^\star$.

Then by Proposition XX; if M_1' is greater than $M_3^\star$ then M_4' is greater than $M_6^\star$; if equal, equal; if less, less.

Therefore M_1 is to M_3 the same ratio as M_4 is to M_6.

■

Modern Restatement

If $M_1 : M_2 \cong M_4 : M_5$ and $M_2 : M_3 \cong M_5 : M_6$, then $M_1 : M_3 \cong M_4 : M_6$.

Modern Proof. Since $M_1 : M_2 \cong M_4 : M_5$ and $M_2 : M_3 \cong M_5 : M_6$, then

$$\begin{aligned}
kM_1 : \ell M_2 \cong& kM_4 : \ell M_5 && \text{by Proposition IV,}\\
\ell M_2 : sM_3 \cong& \ell M_5 : sM_6 && \text{by Proposition IV,}\\
kM_1 \succeq sM_3 \Leftrightarrow& kM_4 \succeq sM_6 && \text{by Proposition XX,}\\
M_1 : M_3 \cong& M_4 : M_6 && \text{by Definition 2.2.5.}
\end{aligned}$$

■

3.23 Proposition XXIII

Euclid's Statement

If there be three magnitudes, and others equal to them in multitude, which taken two and two together are in the same ratio, and the proportion of them be perturbed, they will also be in the same ratio ex aequali.

Restatement

If M_1 is to M_2 the same ratio as M_5 is to M_6, and M_2 is to M_3 the same ratio as M_4 is to M_5, then M_1 is to M_3 the same ratio as M_4 is to M_6.

Proof following to Euclid. Take equimultiples M_1',M_2', and M_4' of M_1, M_2, and M_4. Also take equimultiples M_3^*,M_5^*, and M_6^* of M_3, M_5, and M_6.

Since M_1' and M_2' are equimultiples of M_1 and M_2, then M_1 is to M_2 the same ratio as M_1' is to M_2', by Proposition XV. Similarly since M_5^* and M_6^* are equimultiples of M_5 and M_6, then M_5 is to M_6 the same ratio as M_5^* is to M_6^*. But M_1 is to M_2 the same ratio as M_5 is to M_6, so by Proposition XI M_1' is to M_2' the same ratio as M_5^* is to M_6^*.

Since M_2 is to M_3 the same ratio as M_4 is to M_5, then M_2 is to M_4 as M_3 is to M_5, by Proposition XVI. While M_2' and M_4' are equimultiples of M_2 and M_4. Thus M_2 is to M_4 the same ratio as M_2' as M_4'.

Since M_2 is to M_4 as M_3 is to M_5, and M_2 is to M_4 as M_2' as M_4', then M_3 is to M_5 as M_2' as M_4'. Also M_3 is to M_5 the same ratio as M_3^* as M_5^*. Therefore M_2' is to M_4' as M_3^* is to M_5^*.

Now since M_1' is to M_2' the same ratio as M_5^* is to M_6^*, and M_2' is to M_4' as M_3^* is to M_5^*, then by Proposition XXI; if M_1' is greater than M_3^* then M_4' will be greater than M_6^*; if equal, equal; if less, less. Therefore M_1 is to M_3 the same ratio as M_4 is to M_6. ■

Modern Restatement

Suppose $M_1 : M_2 \cong M_5 : M_6$, and $M_2 : M_3 \cong M_4 : M_5$, then $M_1 : M_3 \cong M_4 : M_6$.

Modern Proof. Since $M_1 : M_2 \cong M_5 : M_6$, and $M_2 : M_3 \cong M_4 : M_5$, then

$$\begin{aligned}
M_1 : M_2 \cong& kM_1 : kM_2 && \text{by Proposition XV,} \\
M_5 : M_6 \cong& \ell M_5 : \ell M_6 && \text{by Proposition XV,} \\
kM_1 : kM_2 \cong& \ell M_5 : \ell M_6 && \text{by Proposition XI,} \quad \text{(i)} \\
M_2 : M_4 \cong& M_3 : M_5 && \text{by Proposition XVI,} \\
M_2 : M_4 \cong& kM_2 : kM_4 && \text{by Proposition XV,} \\
M_3 : M_5 \cong& kM_2 : kM_4 && \text{by Proposition XI,} \\
M_3 : M_5 \cong& \ell M_3 : \ell M_6 && \text{by Proposition XV,} \\
kM_2 : kM_4 \cong& \ell M_3 : \ell M_6 && \text{by Proposition XI,} \quad \text{(ii)} \\
kM_1 \succeq \ell M_3 \Leftrightarrow& kM_4 \succeq \ell M_6 && \text{by (i) and (ii) and Proposition XXI.}
\end{aligned}$$

■

3.24 Proposition XXIV

Euclid's Statement

If a first magnitude have to a second the same ratio as a third has to a fourth, and also a fifth have to the second the same ratio as a sixth to the fourth, the first and fifth added together will have to the second the same ratio as the third and sixth have to the fourth.

Restatement

If M_1 is to M_3 the same ratio as M_4 is to M_6, and M_2 is to M_3 the same ratio as M_5 has to M_6, then $M_1 + M_3$ is to M_2 the same ratio as $M_4 + M_5$ is to M_6.

Proof following to Euclid. Since M_2 is to M_3 the same ratio as M_5 is to M_6, then inversely M_3 is to M_2 as M_6 is to M_5. Now since M_1 is to M_3 as M_4 is to M_6, while M_3 is to M_2 as M_6 is to M_5, then by Proposition XXII, M_1 is to M_2 the same ratio as M_4 is to M_5.

Thus $M_1 + M_2$ is to M_2 the same ratio as $M_4 + M_5$ is to M_5, by Proposition XVIII. Since $M_1 + M_2$ is to M_2 the same ratio as $M_4 + M_5$ is to M_5, while M_2 is to M_3 the same ratio as M_5 is to M_6, then by Proposition XXII $M_1 + M_2$ is to M_3 the same ratio as $M_4 + M_5$ is to M_3. ■

Modern Restatement

If $M_1 : M_3 \cong M_4 : M_6$, and $M_2 : M_3 \cong M_5 : M_6$, then $(M_1 + M_3) : M_2 \cong (M_4 + M_5) : M_6$.

Modern Proof. Since

$$\begin{aligned}
M_1 : M_3 &\cong M_4 : M_6 && \text{by hypothesis,}\\
M_3 : M_2 &\cong M_6 : M_5 && \text{by Porism 3.7.1,}\\
M_1 : M_2 &\cong M_4 : M_5 && \text{by Proposition XXII,}\\
(M_1 + M_2) : M_2 &\cong (M_4 + M_5) : M_5 && \text{by Proposition XVIII,}\\
M_2 : M_3 &\cong M_5 : M_6 && \text{by hypothesis,}\\
(M_1 + M_2) : M_3 &\cong (M_4 + M_5) : M_6 && \text{by Proposition XVIII.}
\end{aligned}$$

■

3.25 Proposition XXV

Euclid's Statement

If four magnitudes be proportional the greatest and the least are greater than the remaining two.

Restatement

If M_1 is to M_2 the same ratio as M_3 is to M_4, and M_1 is the largest, while M_4 is the smallest, then the magnitude $M_1 + M_4$ is greater than $M_2 + M_3$.

Proof following to Euclid. Since M_1 is to M_2 as M_3 is to M_4, and M_3 is less than M_1, while M_4 is less than M_2, then $M_1 - M_3$ is to $M_2 - M_4$ the same ratio as M_1 is to M_2 by Proposition XIX. But M_1 is greater than M_2, therefore $M_1 - M_3$ is also greater than $M_2 - M_4$.

Now since $M_1 - M_3$ is greater than $M_2 - M_4$ then $M_1 - M_3 + (M_2 + M_3)$ is greater than $M_2 - M_4 + (M_2 + M_3)$. Therefore the magnitude $M_1 + M_4$ is greater than the magnitude $M_2 + M_3$. ■

Modern Restatement

If $M_1 : M_2 \cong M_3 : M_4$ and M_1 is the greatest while M_4 is the smallest, then $M_1 + M_4 \succ M_2 + M_3$.

Modern Proof. Since M_1 is the greatest of the four, then $M_1 \succ M_3$ and $M_1 \succ M_2$. Also M_4 is the smallest, thus $M_2 \succ M_4$.

$$
\begin{aligned}
M_1 : M_2 &\cong M_3 : M_4 && \text{by hypothesis,} \\
M_1 &\succ M_3 && \text{by hypothesis,} \\
M_2 &\succ M_4 && \text{by hypothesis,} \\
(M_1 - M_3) : (M_2 - M_4) &\cong M_1 : M_2 && \text{by Proposition XIX,} \\
M_1 &\succ M_2 && \text{by hypothesis,} \\
(M_1 - M_3) &\succ (M_2 - M_4) && \text{by Proposition XVI and XIV,} \\
(M_1 - M_3) + (M_3 + M_4) &\succ (M_2 - M_4) + (M_3 + M_4) && \text{by arithmetic,} \\
M_1 + M_4 &\succ M_2 + M_3 && \text{by arithmetic.}
\end{aligned}
$$

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Vita

Mark Robert Stecher Jr. earned his Bachelor of Art in Mathematics from Brewton Parker College in Mount Vernon, Georgia. After graduating, he began teaching high school mathematics. He later taught middle school math. Mark Robert Stecher Jr., also known as Bobby, soon explored the idea of going back to earn a master's degree. He met Dr. James Madden and Dr. Frank Neubrander in the summer of 2002. The next fall he attended Louisiana State University and started working with the dynamic duo on school improvement projects. Since his days at Brewton Parker, Bobby has been interested in the history of Mathematics. His exploration of Euclid's Elements began in the Fall of 2004, where he started a modern interpretation of Euclid's Theory of Ratio and Proportion.